Measure and convergence: special subsets of the real line and their generalizations

PhD dissertation summary

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In my thesis ([35]) I present different notions of special subsets of the real line and their properties, in particular of those related to measure and convergence. We search for answers to some open questions in this subject and we consider generalizations of well-known facts in the case of a larger cardinality, i.e. in the generalized Cantor space 2^κ, for an uncountable cardinal κ, equipped with the topology generated by sets of extensions of partial functions.

In this document, I describe the motivation behind this work and provide a summary of the main results.

1 Motivation and introduction

The ground breaking but classical idea of Georg Cantor ([11]) to distinguish between countable and uncountable subsets of the real line can be considered to be the first of notions which allow to to divide the family of all subsets of the real line into two classes: small sets and bigger sets. Another two classical notions of a similar kind are the notions of meagre sets (i.e. countable unions of nowhere dense sets, introduced in [1], denoted by $\mathcal{M}$) and Lebesgue measure zero sets (i.e. null sets, [5] and [45], denoted by $\mathcal{N}$). In particular, with respect to all three notions the whole real line is not small. Obviously, every countable set is both of measure zero and meagre. On the other hand, one can easily find a partition of the whole line into two sets, one of which is meagre and the other is null.

It is also worth mentioning that the notions related to measure and category bear some duality. In some cases, this duality is straightforwad, but sometimes it fails. This gives an important motivation for my work, which is to look at measure analogues of some well-known notions and properties related to category. For introduction to the theory of measure and category the reader is referred to [61].

Let me also annotate that in many considerations in set theory of the real line it makes no difference which of the spaces: the real line $\mathbb{R}$, the unit interval $\boldsymbol{I}$, the Cantor space 2^ω, or the Baire space ω^ω we take as the underlying space. This is because, if X, Y are any two of those spaces, there exists a homeomorphism $f\colon X \setminus Q_X \to Y \setminus Q_Y$, where Q_X, Q_Y are countable, and $f[N] \in \mathcal{N}$ if and only if $N \in \mathcal{N}$, $f[M] \in \mathcal{M}$ if and only if $M \in \mathcal{M}$ (see e.g. [9]). In particular, this justifies the use the same notation $\mathcal{N}$ for the ideals of null sets in all of those spaces, and $\mathcal{M}$ for the ideals of meagre sets in all of those spaces.

In particular, in my thesis we usually consider **the Cantor space** 2^ω, which can be seen as a countable product of two-point discrete spaces. Therefore, the basic closed open set in 2^ω is determined by a finite sequence $w \in 2^{<\omega}$. It is denoted by $[w]$,

$$[w] = \{f \in 2^\omega : f \restriction \mathrm{len}(w) = w\}.$$

A set $T \subseteq 2^{<\omega}$ is called a **tree** if for all $t \in T$ and $s \subseteq t$, we have $s \in T$. A tree T is pruned if for all $t \in T$, there exists $s \in T$ with $t \subsetneq s$. If P is a closed set in 2^ω, there is a pruned tree $T_P \subseteq 2^{<\omega}$ such that the set of all infinite branches of T_P (usually denoted by $[T_P]$) equals P. If T is a pruned tree, then $[T]$ is perfect if and only if for any $w \in T$, there exist $w', w'' \in T$ such that $w \subseteq w', w \subseteq w''$, but $w' \nsubseteq w''$ and $w'' \nsubseteq w'$. Such a tree is called a **perfect tree**. A finite sequence $w \in T_P$ is called a **branching point** of a perfect set P if $w^\frown 0, w^\frown 1 \in T_P$. The set of all branching points of P is denoted by $\mathrm{Split}(P)$.

Let P be a perfect set in 2^ω and $h_P : 2^\omega \to P$ be the homeomorphism given by the order isomorphism of $2^{<\omega}$ and $\mathrm{Split}(P)$. We call this homeomorphism the **canonical homeomorphism of** P.

Soon, there appeared even more intricate notions of small subsets of the real line. In particular, the classes of perfectly meager sets and universally null sets play an important role. A set is **perfectly meager** if it is meager relative to any perfect set (here denoted by $\mathrm{P}\boldsymbol{\mathcal{M}}$, this concept appeared first in [47]). A set is **universally null** if it is null with respect to any possible finite diffused Borel measure (denoted here by $\mathrm{U}\boldsymbol{\mathcal{N}}$, this property was studied first in [73]).

Those classes were considered to be dual (see [51]), though some differences between them have been observed. For example, the class of universally null sets is closed under taking products (see [51]), but it is consistent with ZFC that this is not the case for perfectly meager sets (see [62] and [65]).

In [79], P. Zakrzewski proved that two other earlier defined (see [24] and [23]) classes of sets, and smaller then $\mathrm{P}\boldsymbol{\mathcal{M}}$, coincide and are dual to $\mathrm{U}\boldsymbol{\mathcal{N}}$. Therefore, he proposed to call this class **universally meagre** sets (denoted by $\mathrm{U}\boldsymbol{\mathcal{M}}$). A set $A \subseteq 2^\omega$ is universally meagre if every Borel isomorphic image of A in 2^ω is meagre.

In the paper [56], the authors introduced a notion of perfectly meager sets in the transitive sense (denoted here by $\mathrm{P}\boldsymbol{\mathcal{M}}'$), which turned out to be stronger than the classic notion of perfectly meager sets. A set $X \subseteq 2^\omega$ is **perfectly meagre in the transitive sense** if for any perfect set P, there exists an F_σ-set $F \supseteq X$ such that for any t, the set $(F+t) \cap P$ is a meager set relative to P. Further properties of $\mathrm{P}\boldsymbol{\mathcal{M}}'$ sets were investigated in [55], [57], [59] and [58], but still there are some open questions related to the properties of this class. This notion was motivated by its relation to the algebraic sums of sets belonging to different classes of small subsets of 2^ω, and by the obvious fact that a set $X \subseteq 2^\omega$ is perfectly meagre if and only if for any perfect set P, there exists an F_σ-set $F \supseteq X$ such that $F \cap P$ is meagre in P.

A set A is called **strongly null** (strongly of measure zero) if for any sequence of positive $\varepsilon_n > 0$, there exists a sequence of open sets $\langle A_n \rangle_{n \in \omega}$, with $\mathrm{diam} A_n < \varepsilon_n$ for $n \in \omega$, and such that $A \subseteq \bigcup_{n \in \omega} A_n$. I denote the class of such sets by $\mathrm{S}\boldsymbol{\mathcal{N}}$. The idea was introduced for the first time in [6], and **Borel conjectured** that all $\mathrm{S}\boldsymbol{\mathcal{N}}$ sets are countable. This hypothesis turned out to be independent from ZFC (see [44]).

Galvin, Mycielski and Solovay (in [22]) proved that a set $A \in \mathrm{S}\boldsymbol{\mathcal{N}}$ (in 2^ω) if and only if for any meagre set B, there exists $t \in 2^\omega$ such that $A \cap (B+t) = \varnothing$. Therefore, one can consider a dual class of sets. A set A is called **strongly meagre** (strongly first category, denoted by $\mathrm{S}\boldsymbol{\mathcal{M}}$) if for any null set B, there exists $t \in 2^\omega$ such that $A \cap (B+t) = \varnothing$.

We shall say that a set $L \subseteq 2^\omega$ is a **κ-Lusin set** if for any meagre set X, $|L \cap X| < \kappa$, but $|L| \geq \kappa$. An $\aleph_1$-Lusin set is simply called a **Lusin set**. This idea was introduced independently in [47] and [49]. The existence of a Lusin set is independent from ZFC. It is easy to see that under CH such a set exists. The same is true if $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \aleph_1$ (see [9]).

Analogously, an uncountable set $S \subseteq 2^\omega$ is a **Sierpiński set** (introduced in [71]) if for any null set X, $S \cap X$ is countable.

The above classes can be seen as two sequences of decreasing families of sets: for category and measure, as shown in the Table 1.

category	P$\mathcal{M}$	$\supseteq$	U$\mathcal{M}$	$\supseteq$	P$\mathcal{M}'$	$\supseteq$	S$\mathcal{M}$	$\supseteq$	Sierpiński sets
measure			U$\mathcal{N}$		$\supseteq$		S$\mathcal{N}$	$\supseteq$	Lusin sets

Table 1: Classes of special subsets of the real line.

Finally, a set A is called **null-additive** ($A \in \mathcal{N}^*$) if for any null set X, $A + X$ is null. A set A is called **meagre-additive** ($A \in \mathcal{M}^*$) if for any meagre set X, $A + X$ is meagre (see e.g. [78] and [4]).

Despite the fact that the theory of special subsets of the real line is relatively well developed, two of the classes related to category were left without their dual measure counterpart. The motivation of the first part of the thesis is to find classes of sets which can play a dual role to those classes.

The other notion considered in the thesis is the notion of convergence of sequences of real functions. Recall that a sequence $\langle f_n \rangle_{n\in\omega}$ of functions $\boldsymbol{I} \to \boldsymbol{I}$ is **pointwise convergent** ($f_n \to f$) on a set $A \subseteq \boldsymbol{I}$ to a function $f\colon \boldsymbol{I} \to \boldsymbol{I}$ if for any $x \in A$, $\lim_{n\to\infty} f_n(x) = f(x)$. In other words, if

$$\forall_{x\in A} \forall_{\varepsilon>0} \exists_{n\in\omega} \forall_{m\geq n} |f_m(x) - f(x)| \leq \varepsilon.$$

If

$$\forall_{\varepsilon>0} \exists_{n\in\omega} \forall_{m\geq n} \forall_{x\in A} |f_m(x) - f(x)| \leq \varepsilon,$$

we say that the sequence $\langle f_n \rangle_{n\in\omega}$ **converges uniformly** on a set $A \subseteq \boldsymbol{I}$ to f ($f_n \rightrightarrows f$).

A sequence $\langle f_n \rangle_{n\in\omega}$ of functions $\boldsymbol{I} \to \boldsymbol{I}$ **converges quasi-normally** (introduced in [13] and again in [8], see also [9]) on a set $A \subseteq \boldsymbol{I}$ ($f_n \xrightarrow{QN} f$) to a function $f\colon \boldsymbol{I} \to \boldsymbol{I}$ if there exists a sequence $\langle \varepsilon_i \rangle_{i\in\omega} \in (0,\infty)^\omega$ such that $\varepsilon_i \to 0$, and

$$\forall_{x\in A} \exists_{n\in\omega} \forall_{m\geq n} |f_m(x) - f(x)| \leq \varepsilon_m.$$

The important part of considerations in this thesis is related to the well-known Egorov's Theorem which relates notions of convergence with measure. Let us recall that the classic **Egorov's Theorem** (originally proved in [16], see also e.g. [61]) states that given a sequence of Lebesgue measurable functions (we restrict our attention to the real functions $\boldsymbol{I} \to \boldsymbol{I}$) which is pointwise convergent on $\boldsymbol{I}$ and $\varepsilon > 0$, one can find a measurable set $A \subseteq \boldsymbol{I}$ with $m(A) \geq 1 - \varepsilon$ such that the sequence converges uniformly on A.

It seems interesting whether it is possible to generalize this theorem. There are two possible ideas of generalization. The first is to consider other notions of convergence of functions. In particular, we can define a notion of convergence of a sequence of functions with respect to a given ideal I on ω. An **ideal** I on a set X is a collection of subsets of X such that

a) if $A \in I$, and $B \subseteq A$, then $B \in I$,

b) if $A, B \in I$, then $A \cup B \in I$,

c) $X \notin I$.

Given an ideal I on ω and a sequence $\langle x_n \rangle_{n\in\omega} \in \mathbb{R}^\omega$ we say that the sequence **converges** to a point $x \in \mathbb{R}$ **with respect to** I ($x_n \to_I x$) if for every $\varepsilon > 0$,

$$\{n \in \omega : |x_n - x| > \varepsilon\} \in I.$$

This idea was introduced in [29], see also [38], and [60].

Notice that, the classical convergence is just the convergence with respect to the ideal $\mathrm{Fin} = [\omega]^{<\omega}$.

Analogously to the classical convergence, we get different notions of convergence of a sequence $\langle f_n \rangle_{n\in\omega}$ of functions $\boldsymbol{I} \to \boldsymbol{I}$ with respect to an ideal I on ω, which were introduced in [2] and [14]:

pointwise ideal, $f_n \to_I f$ if and only if

$$\forall_{\varepsilon>0} \forall_{x\in A} \{n \in \omega : |f_n(x) - f(x)| \geq \varepsilon\} \in I,$$

quasi-normal ideal, $f_n \xrightarrow{QN}_I f$ if and only if there exists a sequence $\langle \varepsilon_i \rangle_{i\in\omega} \in (0,\infty)^\omega$ such that $\varepsilon_i \to_I 0$ and

$$\forall_{x\in A} \{n \in \omega : |f_n(x) - f(x)| \geq \varepsilon_n\} \in I,$$

uniform ideal, $f_n \rightrightarrows_I f$ if and only if

$$\forall_{\varepsilon>0} \exists_{B\in I} \forall_{x\in A} \{n \in \omega : |f_n(x) - f(x)| \geq \varepsilon\} \subseteq B.$$

I^***-pointwise,** $f_n \to_{I^*} f$ if and only if for all $x \in A$, there exists $M = \{m_i : i \in \omega\} \subseteq \omega$, $m_{i+1} > m_i$ for $i \in \omega$ such that $\omega \setminus M \in I$ and $f_{m_i}(x) \to f(x)$,

I^***-quasi-normal,** $f_n \xrightarrow{QN}_{I^*} f$ if and only if there exists $M = \{m_i : i \in \omega\} \subseteq \omega$, $m_{i+1} > m_i$ for $i \in \omega$ such that $\omega \setminus M \in I$ and $f_{m_i} \xrightarrow{QN} f$ on A,

I^***-uniform,** $f_n \rightrightarrows_{I^*} f$ if and only if there exists $M = \{m_i : i \in \omega\} \subseteq \omega$, $m_{i+1} > m_i$ for $i \in \omega$ such that $\omega \setminus M \in I$ and $f_{m_i} \rightrightarrows f$ on A.

Given two notions of convergence with respect to an ideal, we can ask whether the classic Egorov's Theorem holds for those two notions of convergence in the sense of whether the weaker convergence implies the stronger convergence on a subset of arbitrarily large measure. The answer may often be negative as in the case of uniform and pointwise convergence for many analytic P-ideals (see [53, Theorem 3.4]). But one can also consider other types of convergence, e.g. equi-ideal convergence (for definition see [53] and [54]). And, for example, in the case of analytic P-ideal so called weak Egorov's Theorem for ideals (between equi-ideal and pointwise ideal convergence) was proved by N. Mrożek (see [53, Theorem 3.1]).

The measuability assumption in this theorem seems to play an important role. Actually, it is interesting whether one can drop the assumption on measurability of the functions in the classic Egorov's Theorem. A statement which says that given any sequence of functions $\boldsymbol{I} \to \boldsymbol{I}$ which is pointwise convergent and $\varepsilon > 0$, there exists a set $A \subseteq \boldsymbol{I}$ with $m^*(A) \geq 1 - \varepsilon$ such that the sequence converges uniformly on A, is called the **generalized Egorov's statement**. T. Weiss in his manuscript (see [77]) proved that it is independent from ZFC, and this fact was used in [15]. Then R. Pinciroli studied the method of T. Weiss more systematically (see [63]). For example, he related it to some cardinal coefficients: $\mathrm{non}(\mathcal{N})$, $\mathfrak{b}$ and $\mathfrak{d}$. In particular, he proved that $\mathrm{non}(\mathcal{N}) < \mathfrak{b}$ implies that the generalized Egorov's statement holds, but if, for example, $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}$, then it fails.

Therefore the motivation of the next part of my thesis is to put together both ideas, and study the generalized Egorov's statement (without measurability assumptions) in different cases of ideal convergence ([34]).

Since theory of the real line is already relatively well developed, in the recent years the theory of generalized Cantor and Baire spaces was extensively studied (see, e.g. [46], [20], [21], [41], [42], [69], [70], [17], [18], [19] and many other). An important part of the research in this subject is an attempt to transfer the results in the real line to those spaces (the list of open questions can be found in [43]). Despite the rapid development in this theory, I am not aware of any significant research in the subject of special subsets in 2^κ. Known results are related mainly to the ideal of strongly null sets (see [25] and [26]).

Generalizing theory of the special subsets of the real line to the case of generalized Cantor space is the main motivation of the second half my thesis.

We consider the space 2^κ, called **κ-Cantor space (or the generalized Cantor space)**, endowed with so called bounded topology with basis $\{[x] : x \in 2^{<\kappa}\}$, where for $x \in 2^{<\kappa}$,

$$[x] = \{f \in 2^\kappa : f \restriction \mathrm{dom} x = x\}.$$

If we additionally assume that $\kappa^{<\kappa} = \kappa$, this basis has cardinality κ. This assumption proves to be very convenient when considering the generalized Cantor space, and is assumed throughout the thesis, unless stated otherwise (see e.g. [20]).

A set $T \subseteq 2^{<\kappa}$ will be called a **tree** if for all $t \in T$ and $\alpha < \mathrm{len}(t)$, $t \restriction \alpha \in T$ as well. A branch in a tree is a maximal chain in it. For a tree T, let

$$[T]_\kappa = \{x \in 2^\kappa : \forall_{\alpha<\kappa} x \restriction \alpha \in T\}.$$

It is easy to see that A is closed if and only if $A = [T]_\kappa$ for some tree $T \subseteq 2^{<\kappa}$.

A node $s \in T \subseteq 2^{<\kappa}$ will be called a **branching point** of T if $s^\frown 0, s^\frown 1 \in T$.

The family of **κ-Borel sets** is the smallest family of subsets of 2^κ containing all open sets and closed under complementation, and under taking intersections of size κ.

We say that a set is **κ-meagre** if it is a union of at most κ nowhere dense (in the bounded topology) sets. Notice also that the generalization of the Baire category theorem holds, namely 2^κ is not κ-meagre (see [76, Theorem xv]). The family of all κ-meagre sets in 2^κ is denoted by $\mathcal{M}_\kappa$.

Not all the results of theory of the real line can be easily generalized to the case of 2^κ. One of the main obstacles is the notion of compactness. We shall say that a topological space X is **κ-compact** (or κ-Lindelöf) if every open cover of X has a subcover of cardinality less than κ (see [52], [27]). Obviously, the Cantor space 2^ω is ω-compact (i.e. compact in the traditional sense). But it is not always the case that 2^κ is κ-compact. Recall that a cardinal number κ is weakly compact if it is uncountable and for every two-colour colouring of the set of all two-element subsets of κ, there exists a set $H \subseteq \kappa$

of cardinality κ, which is homogeneous (every two-element subset of H have the same colour in the considered colouring) (see [28]). Recall that every weakly compact cardinal is strongly inaccessible. Actually, the generalized Cantor space 2^κ is κ-compact if and only if κ is a weakly compact cardinal (see [52]). And there is even more to that. In particular, the generalized Cantor space 2^κ and the generalized Baire spaces κ^κ are homeomorphic if and only if κ is not a weakly compact cardinal.

The other major difference concerns the notion of perfect set. A set $P \subseteq 2^\kappa$ is a **perfect set** if it is closed and has no isolated points. A tree $T \subseteq 2^{<\kappa}$ is **perfect** if for any $t \in T$, there exists $s \in T$ such that $t \subseteq s$ and $s \in \mathrm{Split}(T)$. Notice that a set $P \subseteq 2^\kappa$ is perfect if and only if T_P is a perfect tree.

But a perfect tree T will be called **κ-perfect** if for every limit $\beta < \kappa$, and $t \in 2^\beta$ such that $t{\restriction}\alpha \in T$, we have $t \in T$. Notice that every κ-perfect tree is order-isomorphic with $2^{<\kappa}$. A set $P \subseteq 2^\kappa$ is **κ-perfect** if $P = [T]_\kappa$ for a κ-perfect tree T. Obviously, every κ-perfect set is perfect. On the other hand the converse does not hold.

In 2^ω every uncountable analytic set contains a perfect set. On the other hand, the generalization of this theorem for 2^κ may not be true even for closed sets. There may even exist a perfect set which does not contain a κ-perfect set.

One can also study different notions of convergence of functions $2^\kappa \to 2^\kappa$, and consider notions of special subsets related to convergence. Finally, we may also consider the possibility of introducing Egorov's Theorem in 2^κ.

2 Structure of the thesis

My thesis consists of two main parts:

(a) the developments in the real line (Chapters 2–3),

Chapter 2. We discuss special subsets of the Cantor space 2^ω. The theory of special subsets is already well developed (see above). I introduce two notions of such sets, which were not considered before: the class of perfectly null sets and the class of sets which are perfectly null in the transitive sense ([36]). These classes may play the role of duals on the measure side to the corresponding classes on the category side. We investigate their properties, and although the main problem of whether the classes of perfectly null sets and universally null sets are consistently different remains open, we prove some results related to this question and study their version on the category side.

Chapter 3. We study problems related to Egorov's Theorem, which describes a relation between convergence and measure. Egorov's Theorem can be generalized to some notions of ideal convergences (see e.g. [53]), and T. Weiss has proven ([77]) that the generalized Egorov's statement (i.e. the theorem without the assumption on measurability) is independent from ZFC. Integrating both ideas, we prove that the generalized Egorov's statement as well as its negation are consistent with ZFC in different cases of ideal convergence ([34]).

(b) generalization to 2^κ (Chapters 4-8).

Chapter 4. Many of the classical notions of special subsets of 2^ω can be considered in the case of generalized Cantor space 2^κ. Although the theory of generalized

Cantor space 2^κ has recently been vastly developed (see e.g. [43]), the theory of special subsets of 2^κ seems to be largely omitted from those considerations. We study those classes of sets in this setting ([37]). It turns out that many of properties of subsets of 2^ω can be easily proved in 2^κ, although sometimes one has to use some additional set-theoretic assumptions.

Chapter 5. We deal with less common classes of small sets in 2^κ.

Chapter 6. I present different types of convergence of κ-sequences of functions $2^\kappa \to 2^\kappa$, and study properties of special subsets of 2^κ related to the notion of convergence ([33]). We relate those properties to the sequence selection principles.

Chapter 7. We consider convergence of sequences of points and functions with respect to an ideal on κ.

Chapter 8. Finally, to relate measure and convergence properties in 2^κ, we study the possibility of introducing Egorov's Theorem in 2^κ. Since no method of constructing measure in 2^κ which fulfils all reasonable requirements is known, we consider the properties such set-function should have to enable the proof of Egorov's Theorem. I leave the question of existence of such a function which satisfies some additional reasonable conditions open. Every κ-strongly null set is null with respect to such a set function which satisfies some additional properties. We study also the ideal version of Egorov's Theorem in 2^κ.

Chapter 1 consists of introduction and preliminaries, and Chapter 9 contains conclusions, comments on further development and open problems.

3 Overview of the results

3.1 The real line

The theory of special subsets of the real line and the theory of convergence of sequences of real functions are relatively well developed. In my thesis I present some further developments is two subjects: perfectly null sets (Chapter 2) and generalized Egorov's statement for ideals (Chapter 3).

The idea of constructing perfectly null sets comes from the observed duality between measure and category and the lack of notion dual to the class of perfectly meagre set. We construct such a notion.

We start by defining a canonical measure on a perfect set $P \subseteq 2^\kappa$. Let $A \subseteq P$ be such that $h_P^{-1}[A]$ is measurable in 2^ω, where $h_P\colon 2^\omega \to P$ is the canonical homeomorphism on P. We define

$$\mu_P(A) = m(h_P^{-1}[A]).$$

Measure μ_P will be called **the canonical measure on** P. A set A such that $\mu_P(A\cap P) = 0$ will be called P**-null**.

We shall say that $A \subseteq 2^\omega$ is **perfectly null** if it is P-null for any perfect set $P \subseteq 2^\omega$. The class of perfectly null sets is denoted by $\mathrm{P}\boldsymbol{\mathcal{N}}$.

We study properties of this class. Nevertheless, the answer to the main problem in this chapter of whether it is consistent with ZFC that there exists a perfectly null set which is not universally null remains unclear. In particular, we do not know whether the class of perfectly null sets is closed under taking products.

All known solutions to the dual problem use the idea of Lusin function or ideas similar to it. The **Lusin function** $\mathcal{L}: \omega^\omega \to 2^\omega$ is a continuous one-to-one function with measurable inverse such that if L is a Lusin set, then $\mathcal{L}[L]$ is perfectly meager. It was defined in [48], and extensively described in [72].

Pursuing the answer to the above problem, we have shown that if there exists a measure analogue of the Lusin function it cannot be constructed in an analogous way.

Proposition 1 *Let $\mathcal{S}: \omega^\omega \to 2^\omega$ be a function such that there exists a sequence $\langle P_s : s \in \omega^{<\omega}\rangle$ such that for $s \in \omega^{<\omega}$, $P_s \subseteq 2^\omega$ is a perfect set, and for $n, m \in \omega$:*

(a) $n \neq m \Rightarrow P_{s^\frown n} \cap P_{s^\frown m} = \varnothing$,

(b) $P_{s^\frown n} \subseteq P_s$,

(c) $diam(P_s) \leq 1/2^{len(s)}$,

and $\mathcal{S}(x)$ is the only element of $\bigcap_{n\in\omega} P_{x\restriction n}$. Then there exists a perfect set $Q \subseteq 2^\omega$ such that

$$m\left(\mathcal{S}^{-1}[\bigcup\{P_s : s \in \omega^{<\omega} \wedge \mu_Q(P_s) = 0\}]\right) < 1.$$

In the above proposition we equip ω^ω with a measure m such that

$$m([w]) = \prod_{i=0}^{\mathrm{len}(w)-1} \frac{1}{2^{w(i)+1}},$$

where $w \in \omega^{<\omega}$.

Also, if the class P$\mathcal{N}$ is closed under homeomorphisms, then U$\mathcal{N}$ = P$\mathcal{N}$.

Theorem 2 *If the class P$\mathcal{N}$ is closed under homeomorphisms of 2^ω, then U$\mathcal{N}$ = P$\mathcal{N}$.*

Finally, we have considered some simpler classes of perfect subsets in which analogous problems can be at least partially solved.

A branching point is on level $i \in \omega$ if there exist i branching points below it. The set of all branching points of P on level i will be denoted by $\mathrm{Split}_i(P)$. Let

$$s_i(P) = \min\{\mathrm{len}(w) : w \in \mathrm{Split}_i(P)\}$$

and

$$S_i(P) = \max\{\mathrm{len}(w) : w \in \mathrm{Split}_i(P)\}.$$

A perfect set P will be called a **balanced perfect** set if $s_{i+1}(P) > S_i(P)$ for all $i \in \omega$. This definition generalizes the notion of uniformly perfect set, which can be found in [7].

A perfect set P is **uniformly perfect** if for any $i \in \omega$, either $2^i \cap T_P \subseteq \mathrm{Split}(P)$ or $2^i \cap \mathrm{Split}(P) = \varnothing$.

A set that is null in any balanced (respectively, uniformly) perfect set will be called **balanced perfectly null** (respectively, **uniformly perfectly null**). The class of such sets will be denoted by bP$\mathcal{N}$ (respectively, uP$\mathcal{N}$).

We prove the following theorem, which bears some resemblance to the result of Recław ([65]).

Theorem 3 *If there exists a Sierpiński set, then there are $X, Y \in$ bP$\mathcal{N}$ such that $X \times Y \notin$ uP$\mathcal{N}$.*

Next we study analogues to the small sets considered by Bartoszyński (in [4]) with respect to the canonical measure on perfect sets. Two approaches are presented, and the second one seems to be more promising. In this approach the definition of sets which are small in P is quite technical and therefore is omitted here. In particular, we have shown that every set which is null in a perfect set P can be presented as a union of two small sets in P.

Proposition 4 *If $X \subseteq P$ is P-null, then $X \subseteq A_1 \cup A_2$, where A_1, A_2 are small in P.*

We also study additive properties of small sets in P.

Proposition 5 *Let $X \subseteq P$ be a small set in P and Y be a additively null set. Then $X + Y$ is P-null.*

Finally, we construct an analogue (P$\mathcal{N}'$) of the class of perfectly meagre in the transitive sense sets (P$\mathcal{M}'$). We call a set X **perfectly null in the transitive sense** if for any perfect set P, there exists a G_δ set $G \supseteq X$ such that for any t, the set $(G + t) \cap P$ is P-null.

It is known that every strongly meagre set is P$\mathcal{M}'$, and every P$\mathcal{M}'$ set is universally meagre, and that it is consistent with ZFC that those inclusions are proper. We prove some of the analogous results on the measure side.

Theorem 6 *Every strongly null set is perfectly null in the transitive sense.*

Under certain set-theoretical assumptions, there exists a universally null set which is not P$\mathcal{N}'$.

Theorem 7 *If there exists a universally null set of cardinality $\mathfrak{c}$, then there exists $Y \in U\mathcal{N} \setminus bP\mathcal{N}' \subseteq U\mathcal{N} \setminus P\mathcal{N}'$.*

The other two problems of whether there exists a P$\mathcal{N}'$ set which is not S$\mathcal{N}$, and whether every P$\mathcal{N}'$ set is U$\mathcal{N}$ remain open.

The consideration of P$\mathcal{M}'$ class started with its additive properties. Hence, we study the additive properties of P$\mathcal{N}'$ sets as well.

Theorem 8 *Let $X \in P\mathcal{N}'$, and let Y be an $SR^{\mathcal{N}}$ set. Then $X + Y \in P\mathcal{N}$.*

In the above theorem, a set Y is a $SR^{\mathcal{N}}$ set if for every Borel set $H \subseteq 2^\omega \times 2^\omega$ such that $H_x = \{y \in 2^\omega : \langle x, y\rangle \in H\}$ is null for any $x \in 2^\omega$, $\bigcup_{x \in Y} H_x$ is null as well ([3]).

Nevertheless, the main problem of whether if $A \in$ S$\mathcal{M}$, and $B \in$ P$\mathcal{N}'$, then $A + B$ an s_0-set remains unsolved. A set A is an s_0**-set** if for every perfect set P, there exists $Q \subseteq P$ such that $A \cap Q = \varnothing$ (see [50]).

In Chapter 3 we study the second subject in set theory of real line, which concerns generalizations of Egorov's Theorem. Previously, it has been known that the Egorov's Theorem without the assumption on measurability (so called generalized Egorov's statement) is consistent with ZFC (see [77]), as is its negation. Also the ideal version of Egorov's Theorem (with the measurability assumption) was studied for different notions of ideal convergence (see e.g. [53]). Therefore, we study the generalized Egorov's statement in the case of different notions of ideal convergence.

We start with generalizing the method of Pinciroli (see [63]). For a sequence of functions $f_n : \boldsymbol{I} \to \boldsymbol{I}$ and subsets $A \subseteq \boldsymbol{I}$, we consider a notion of convergence $f_n \looparrowright f$ on A. Let $\mathcal{F} \subseteq \{\langle f_n\rangle_{n\in\omega} : \forall_{n\in\omega} f_n : \boldsymbol{I} \to \boldsymbol{I}\}$ be an arbitrary family of sequences of functions.

We consider two hypotheses between $\mathcal{F}$ and $\looparrowright$:

($H^{\Rightarrow}(\mathcal{F},\looparrowright)$) There exists $o:\mathcal{F}\to(\omega^\omega)^{\boldsymbol{I}}$ such that for every $F\in\mathcal{F}$ and every $A\subseteq\boldsymbol{I}$ if $o(F)[A]$ is bounded in $(\omega^\omega,\leq)$, then $F\looparrowright 0$ on A.

($H^{\Leftarrow}(\mathcal{F},\looparrowright)$) There exists cofinal $o:\mathcal{F}\to(\omega^\omega)^{\boldsymbol{I}}$ such that for every $F\in\mathcal{F}$ and every $A\subseteq\boldsymbol{I}$, if $F\looparrowright 0$ on A, then $o(F)[A]$ is bounded in $(\omega^\omega,\leq)$.

We have the following theorems.

Theorem 9 *Assume that* $non(\boldsymbol{\mathcal{N}})<\mathfrak{b}$, *and* $H^{\Rightarrow}(\mathcal{F},\looparrowright)$. *Then for any* $\langle f_n\rangle_{n\in\omega}\in\mathcal{F}$ *and any* $\varepsilon>0$, *there exists* $A\subseteq\boldsymbol{I}$ *such that* $m^*(A)\geq 1-\varepsilon$ *and* $f_n\looparrowright 0$ *on* A.

Theorem 10 *Assume that* $non(\boldsymbol{\mathcal{N}})=\mathfrak{c}$, *and that there exists a* $\mathfrak{c}$*-Lusin set. If* $H^{\Leftarrow}(\mathcal{F},\looparrowright)$ *holds, then there exist* $\langle f_n\rangle_{n\in\omega}\in\mathcal{F}$ *and* $\varepsilon>0$ *such that for all* $A\subseteq\boldsymbol{I}$ *with* $m^*(A)\geq 1-\varepsilon$, $f_n\not\looparrowright 0$ *on* A.

We prove that both the ideal version of the generalized Egorov's statement and its negation are consistent with ZFC with respect to different notions of ideal convergence.

Theorem 11 *The ideal version of the generalized Egorov's statement and its negation are consistent with ZFC with respect to following notions of convergence:*

(a) between pointwise and equi-convergence with respect to analytic P-ideals,

(b) between pointwise and uniform convergence with respect to countably generated ideals,

(c) between pointwise-I^* *and uniform-*I^* *convergence with respect to countably generated ideals,*

(d) between pointwise and uniform convergence with respect to ideals of form Fin^α, $\alpha<\omega_1$.

Additionally, I prove Egorov's Theorem (with measurability assumption) in case in which it was not proven before ((b) and (c)).

Theorem 12 *If* $I\subseteq 2^\omega$ *is a countably generated ideal, and* $f_n\colon\boldsymbol{I}\to\boldsymbol{I}$, $n\in\omega$ *are Lebesgue-measurable functions such that* $f_n\to_I 0$ *and* $\varepsilon>0$, *then there exists a measurable set* $B\subseteq\boldsymbol{I}$ *such that* $m(B)\leq\varepsilon$ *and* $f_n\rightrightarrows_I 0$ *on* $\boldsymbol{I}\smallsetminus B$.

Theorem 13 *If* $I\subseteq 2^\omega$ *is a countably generated ideal and* $f_n\colon\boldsymbol{I}\to\boldsymbol{I}$, $n\in\omega$ *are Lebesgue-measurable functions such that* $f_n\to_{I^*} 0$ *and* $\varepsilon>0$, *then there exists a measurable set* $B\subseteq\boldsymbol{I}$ *such that* $m(B)\leq\varepsilon$ *and* $f_n\rightrightarrows_{I^*} 0$ *on* $\boldsymbol{I}\smallsetminus B$.

This generalization of Pinciroli method provides combinatorial properties ($H^{\Rightarrow}(\mathcal{F},\looparrowright)$) and ($H^{\Leftarrow}(\mathcal{F},\looparrowright)$), which were mentioned above, and which imply that the generalized Egorov's statement (respectively, its negation) is consistent with ZFC. Later on, M. Repický ([67]) further generalized my results by studying the closure properties of classes of ideals satisfying those properties. He also introduced an analogous property ($M^{\Rightarrow}(\mathcal{F},\looparrowright)$) which implies that the Egorov's Theorem (with measurability assumption) holds for convergence with respect to such ideal.

Nevertheless, the research in this subject needs to be continued to answer some further open problems.

3.2 In the generalized Cantor space

In the subsequent chapters of the thesis we study the generalized Cantor space 2^κ, where κ is an uncountable regular cardinal.

In Chapter 4 we introduce simple notions of special subsets in 2^κ, which are generalizations of the notions summarized in [51] and [9]:

(a) λ-κ-Lusin sets. A set $L \subseteq 2^\kappa$ such that $|L| \geq \lambda$, and if $X \subseteq 2^\kappa$ is any κ-meager set, then $|X \cap L| < \lambda$ will be called a **λ-κ-Lusin set**. A κ^+-κ-Lusin set is simply called a **Lusin set for** κ. We prove that such a set exists under certain set-theoretic assumptions.

Theorem 14 *If $\lambda = cov(\mathcal{M}_\kappa) = cof(\mathcal{M}_\kappa)$, then there exists a λ-κ-Lusin set.*

In particular, it exists under CH_κ.

(b) κ-strongly measure zero sets. A set $A \subseteq 2^\kappa$ will be called **κ-strongly measure zero** ($S\mathcal{N}_\kappa$) if for every $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$, there exists $\langle x_\alpha \rangle_{\alpha<\kappa}$ such that $x_\alpha \in 2^{\xi_\alpha}$, $\alpha < \kappa$ and $A \subseteq \bigcup_{\alpha<\kappa} [x_\alpha]$ (see also [25] and [26]). We prove that one of the implications analogous to Galvin-Mycielski-Soloway Theorem holds.

Proposition 15 *Let A be such that for any nowhere dense set F, there exists $x \in 2^\kappa$ such that $(x + A) \cap F = \varnothing$. Then, A is $S\mathcal{N}_\kappa$.*

The other implication was proven when κ is weakly compact.

Theorem 16 *Assume that κ is a weakly compact cardinal, and $A \subseteq 2^\kappa$ is $S\mathcal{N}_\kappa$. Then for any κ-meagre set F, there exists $x \in 2^\kappa$ such that $(x + A) \cap F = \varnothing$.*

(c) κ^+-concentrated sets (a set $A \subseteq 2^\kappa$ will be called **λ-concentrated on a set** $B \subseteq 2^\kappa$ for $\kappa < \lambda \leq 2^\kappa$ if for any open set G such that $B \subseteq G$, we have $|A \setminus G| < \lambda$), and we prove the following inclusions related to this class.

Proposition 17 *A set $A \subseteq 2^\kappa$ is a Lusin set for κ if and only if $|A| > \kappa$ and is κ^+-concentrated on every dense set $D \subseteq 2^\kappa$ with $|D| = \kappa$.*

Proposition 18 *If a set $A \subseteq 2^\kappa$ is κ^+-concentrated on a set B such that $|B| \leq \kappa$, then $A \in S\mathcal{N}_\kappa$.*

Thus, every Lusin set for κ is κ-strongly null.

(d) κ-perfectly κ-meagre (a set $A \subseteq 2^\kappa$ will be called **κ-perfectly κ-meagre** ($P_{\boldsymbol{\kappa}}\mathcal{M}_\kappa$) if for every κ-perfect $P \subseteq 2^\kappa$, $A \cap P$ is κ-meagre relatively to P), perfectly κ-meagre (a set $A \subseteq 2^\kappa$ will be called **perfectly κ-meagre** ($P\mathcal{M}_\kappa$) if for every perfect $P \subseteq 2^\kappa$, $A \cap P$ is κ-meagre relatively to P), and κ-λ sets (a set $A \subseteq 2^\kappa$ is a **κ-λ-set** if for any $B \subseteq A$ with $|B| \leq \kappa$ there exists a sequence $\langle B_\alpha \rangle_{\alpha<\kappa}$, where $B_\alpha \subseteq 2^\kappa$ are open, and $\bigcap_{\alpha<\kappa} B_\alpha \cap A = B$), and among other properties we prove the following.

Proposition 19 *Every κ-λ-set $A \subseteq 2^\kappa$ is perfectly κ-meagre.*

On the other hand we do not know if there exists perfectly κ-meagre set of cardinality bigger than κ in every model of ZFC.

(e) κ-σ-sets. A set $A \subseteq 2^\kappa$ will be called **κ-σ-set** if for any sequence of closed sets $\langle F_\alpha\rangle_{\alpha<\kappa}$, there exists a sequence of open sets $\langle G_\alpha\rangle_{\alpha<\kappa}$ such that

$$A \cap \bigcup_{\alpha<\kappa} F_\alpha = A \cap \bigcap_{\alpha<\kappa} G_\alpha.$$

We prove that every such set is perfectly κ-meagre,

(f) κ-Q-sets. A set $A \subseteq 2^\kappa$ will be called **κ-Q-set** if for any set $B \subseteq A$, there exists a sequence of closed sets $\langle F_\alpha\rangle_{\alpha<\kappa}$ such that

$$A \cap \bigcup_{\alpha<\kappa} F_\alpha = B.$$

Obviously, every κ-Q-set is a κ-σ-set.

We also study the generalization of selection properties in κ (the first systematic study of selection principles on the real line can be found in [68]).

A family of open subsets $\mathcal{U}$ of a set X will be called a **κ-cover** of X if for any $A \in [X]^{<\kappa}$ there exists $U \in \mathcal{U}$ such that $A \subseteq U$. It is a **γ-κ-cover** if $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$, and

$$X \subseteq \bigcup_{\alpha<\kappa} \bigcap_{\alpha<\beta<\kappa} U_\beta.$$

The family of all κ-covers of X will be denoted by $\Omega_\kappa(X)$, and the family of all κ-γ-covers will be denoted by $\Gamma_\kappa(X)$. The family of all open covers of size κ of X, is denoted by $\mathcal{O}_\kappa(X)$. The underlying set can be omitted in this notation if it is apparent from the context. We always assume that the covers which are considered are proper, i.e. the set itself is never an element of its cover.

$X \subseteq 2^\kappa$ will be called a **κ-γ-set** if for every open κ-cover $\mathcal{U}$ of X there exists a sequence $\langle U_\alpha\rangle_{\alpha<\kappa} \in \mathcal{U}^\kappa$ such that $\{U_\alpha : \alpha < \kappa\}$ is a κ-γ-cover.

If $\mathcal{A}, \mathcal{B}$ are families of open covers of a set X, we shall say that it has **$S_1^\kappa(\mathcal{A}, \mathcal{B})$ property** if for every sequence $\langle \mathcal{U}_\alpha\rangle_{\alpha<\kappa} \in \mathcal{A}^\kappa$, there exists a sequence $\langle U_\alpha\rangle_{\alpha<\kappa}$ such that $U_\alpha \in \mathcal{U}_\alpha$, for all $\alpha < \kappa$, and $\{U_\alpha : \alpha < \kappa\} \in \mathcal{B}$.

In particular, an we proved the following.

Theorem 20 *A set $X \subseteq 2^\kappa$, with $|X| \geq \kappa$ is a κ-γ-set if and only if it has $S_1^\kappa(\Omega_\kappa, \Gamma_\kappa)$ property.*

A cover $\mathcal{U}$ of a set X is **essentially of size κ** if for every $\mathcal{V} \in [\mathcal{U}]^{<\kappa}$, $X \setminus \bigcup \mathcal{V} \neq \emptyset$.

We will say that a set X satisfies **$U_{<\kappa}^\kappa(\mathcal{A}, \mathcal{B})$ principle** if for every sequence $\langle \mathcal{U}_\alpha\rangle_{\alpha<\kappa} \in \mathcal{A}^\kappa$ of covers essentially of size κ, there exists $\langle \mathcal{V}_\alpha\rangle_{\alpha<\kappa}$ such that $\mathcal{V}_\alpha \in [\mathcal{U}_\alpha]^{<\kappa}$ for all $\alpha < \kappa$, and $\{\bigcup \mathcal{V}_\alpha : \alpha < \kappa\} \in \mathcal{B}$.

A set X has **κ-Hurewicz property** if it satisfies $U_{<\kappa}^\kappa(\mathcal{O}_\kappa, \Gamma_\kappa)$ principle.

Proposition 21 *If X satisfies $S_1^\kappa(\Gamma_\kappa, \Gamma_\kappa)$, then it has κ-Hurewicz property.*

Hence, every κ-γ-set has κ-Hurewicz property. On the other hand, we proved that every λ-κ-Lusin set does not have this property.

Proposition 22 *If $\kappa < \lambda \le 2^\kappa$, and $L \subseteq 2^\kappa$ is a λ-κ-Lusin set, then L does not have κ-Hurewicz property.*

On the other hand, it does have κ-Menger property. A set has **κ-Menger property** if it satisfies $U^{\kappa}_{<\kappa}(\mathcal{O}_\kappa, \mathcal{O}_\kappa)$ principle.

Proposition 23 *Let $L \subseteq 2^\kappa$ be a Lusin set for κ. Then L has κ-Menger property.*

Obviously, if a set has κ-Rothberger property, it is κ-strongly null.

Proposition 24 *If $A \subseteq 2^\kappa$ has κ-Rothberger property, then $A \in S\mathcal{N}_\kappa$.*

But we even have the following.

Proposition 25 *If $A \subseteq 2^\kappa$ is κ^+-concentrated on a set $B \subseteq 2^\kappa$ with $|B| \le \kappa$, then A has κ-Rothberger property.*

Hence, the whole 2^κ does not have this property.
On the other hand, we have the following.

Theorem 26 *Every κ-γ-set of cardinality $\ge \kappa$ has κ-Rothberger property.*

In particular, the whole space 2^κ cannot be a κ-γ-set.
In Chapter 5, we study in 2^κ versions of less common notions of special subsets. We introduce:

(a) X-small sets, which follow the idea of small sets in $\omega_1^{\omega_1}$ presented in [25]. If $X \subseteq \kappa$, then a set $A \subseteq 2^\kappa$ will be called **X-small** if there exists $\langle a_\alpha \rangle_{\alpha \in X} \in (2^\kappa)^X$ such that

$$A \subseteq \bigcup_{\alpha \in X} [a_\alpha \restriction \alpha].$$

Notice that A is S$\mathcal{N}_\kappa$ if it is X-small for any $X \in [\kappa]^\kappa$. Let $\lambda < \kappa$. We say that a set $A \subseteq 2^\kappa$ is **λ-X-small** for $X \subseteq \kappa$ if there exists $\langle a_{\alpha,\beta} \rangle_{\alpha \in X, \beta < \lambda} \in ((2^\kappa)^X)^\lambda$ such that

$$A \subseteq \bigcup_{\alpha \in X} \bigcup_{\beta < \lambda} [a_{\alpha,\beta} \restriction \alpha].$$

We prove the following properties related to this class of sets.

Proposition 27 *Let $A \subseteq 2^\kappa$ be small in 2^κ. Then $A \in S\mathcal{N}_\kappa$.*

Proposition 28 *Assume $\diamondsuit_\kappa$. If C is a closed unbounded set in κ, then 2^κ is C-small.*

Proposition 29 *Assume $V = L$. Then 2^κ is X-small for every stationary set $X \subseteq \kappa$.*

We also show that every set small in 2^κ is nowhere dense.

Proposition 30 *Every set which is small in 2^κ is nowhere dense.*

But the reversed implication does not hold.

Proposition 31 *There exists a nowhere dense set $A \subseteq 2^\kappa$ which is not κ-strongly null.*

(b) κ-meagre additive sets (a set $A \subseteq 2^\kappa$ will be called **κ-meagre additive** if for any κ-meagre set F, $A + F$ is κ-meagre), and we prove a combinatorial characterization of κ-meagre additive sets for strongly inaccessible κ.

Proposition 32 *Assume that κ is strongly inaccessible, and $X \subseteq 2^\kappa$. Then X is κ-meagre additive if and only if for every increasing sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ there exist a sequence $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ and $z \in 2^\kappa$ such that*

$$X \subseteq \left\{x \in 2^\kappa \colon \exists_{\alpha<\kappa} \forall_{\alpha<\beta<\kappa} \exists_{\gamma<\kappa} \left(\eta_\beta \le \xi_\gamma < \xi_{\gamma+1} \le \eta_{\beta+1} \wedge \forall_{\xi_\gamma \le \delta < \xi_{\gamma+1}} x(\delta) = z(\delta)\right)\right\}.$$

This characterization implies that every κ-meagre additive set is κ-perfectly κ-meagre.

Proposition 33 *Assume that κ is a strongly inaccessible cardinal. Then every κ-meagre additive set is κ-perfectly κ-meagre.*

(c) κ-Ramsey null sets (a generalization of the concept of Ramsey-null sets introduced in [64]). For $\alpha < \kappa$, $s \in 2^\alpha$ and $S \in [\kappa \setminus \alpha]^\kappa$, let

$$[s, S] = \{x \in 2^\kappa \colon s^{-1}[\{1\}] \subseteq x^{-1}[\{1\}] \subseteq s^{-1}[\{1\}] \cup S \wedge |x^{-1}[\{1\}] \cap S| = \kappa\}.$$

A set $A \subseteq 2^\kappa$ will be called **κ-Ramsey null ($\kappa - CR_0$)** if for any $\alpha < \kappa$, $s \in 2^\alpha$ and $S \in [\kappa \setminus \alpha]^\kappa$, there exists $S' \in [S]^\kappa$ such that $[s, S'] \cap A = \varnothing$.

In particular, we prove the following.

Proposition 34 *Assume that κ is a weakly inaccessible cardinal. Then every κ-γ-set which is not closed in 2^κ is κ-Ramsey null.*

On the other hand, we have not been able to determine the additivity of the ideal of κ-Ramsey null sets.

(d) κ-T'-sets (a generaliztion of notion of T'-sets considered in [55], and also [66]). A set $A \subseteq 2^\kappa$ is here called a **κ-T'-set** if there exists a sequence of cardinal numbers $\langle \lambda_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that for every increasing sequence $\langle \delta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ with $\delta_0 = 0$, and $\delta_\alpha = \bigcup_{\beta<\alpha} \delta_\beta$ for limit α, there exists a sequence $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$, and

$$H_\alpha \in \left[2^{\delta_{\eta_\alpha+1} \setminus \delta_{\eta_\alpha}}\right]^{\le \lambda_{\eta_\alpha}},$$

for all $\alpha < \kappa$ such that

$$A \subseteq \{x \in 2^\kappa \colon \forall_{\beta<\kappa} \exists_{\beta<\alpha<\kappa} x \restriction (\delta_{\eta_\alpha+1} \setminus \delta_{\eta_\alpha}) \in H_\alpha\}.$$

We prove various characterizations of this notion. The class of κ-T'-sets forms an ideal.

Proposition 35 *Assume that κ is a weakly inaccessible cardinal. The class of κ-T'-sets forms a κ^+-complete ideal of subsets of 2^κ.*

Moreover, and an algebraic sum of two κ-T'-sets is still a κ-T'-set.

Proposition 36 *If $A, B \subseteq 2^\kappa$ are κ-T'-sets, then $A + B$ is also a κ-T'-set.*

We get also the following relation with other classes of special subsets.

Proposition 37 *Assume that κ is a strongly inaccessible cardinal. Then every κ-γ-set is a κ-T'-set.*

Proposition 38 *Assume that κ is a strongly inaccessible cardinal. Then every κ-T'-set is κ-meagre additive.*

Thus, if κ is strongly inaccessible, every κ-γ-set is κ-meagre additive. Under some additional assumptions this inclusion cannot be reversed.

(e) κ-v_0-sets (a generaliztion of the notion of v_0-sets, which were studied in [39]). A κ-perfect set P is a **κ-Silver perfect** if for all $\alpha < \kappa$ and any $i \in \{0, 1\}$,

$$\exists_{s \in 2^\alpha \cap T_P} s^\frown i \in T_P \Rightarrow \forall_{s \in 2^\alpha \cap T_P} s^\frown i \in T_P.$$

A set $A \subseteq 2^\kappa$ is a **κ-v_0-set** if for all κ-Silver perfect set $P \subseteq 2^\kappa$, there exists a κ-Silver perfect set $Q \subseteq P$ such that $A \cap Q = \varnothing$. The notion of κ-v_0 sets was considered in [42].

In my thesis we study the relation between this notion and other notions of special subsets of 2^κ.

Proposition 39 *Assume that κ is a strongly inaccessible cardinal. Then every κ-perfectly κ-meagre set in 2^κ is a κ-v_0-set.*

Proposition 40 *Every κ-strongly null set in 2^κ is a κ-v_0-set.*

We have left as a subject for further research the classes of κ-l_0-sets and κ-m_0-set related to Laver and Miller forcings respectively (and are generaliztions of notions considered in [40])).

In Chapter 6, we introduce and study the convergence of κ-sequences of functions $2^\kappa \to 2^\kappa$. We consider κ-uniform convergence, κ-quasi normal convergence, and κ-pointwise convergence.

A sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ is **κ-pointwise convergent** to a function $f : 2^\kappa \to 2^\kappa$ (denoted by $f_\alpha \to_\kappa f$) on $A \subseteq 2^\kappa$ if

$$\forall_{x \in A} \forall_{\beta<\kappa} \exists_{\gamma<\kappa} \forall_{\gamma \le \alpha < \kappa} f_\alpha(x) \in [f(x) \restriction \beta].$$

Similarly, we say that such a sequence of functions **converges κ-uniformly** to $f : 2^\kappa \to 2^\kappa$ (denoted by $f_\alpha \rightrightarrows_\kappa f$) on $A \subseteq 2^\kappa$ if

$$\forall_{\beta<\kappa} \exists_{\gamma<\kappa} \forall_{x \in A} \forall_{\gamma \le \alpha < \kappa} f_\alpha(x) \in [f(x) \restriction \beta].$$

Finally, we say that a sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ **converges κ-quasi-normally** to a function $f : 2^\kappa \to 2^\kappa$ (denoted by $f_\alpha \xrightarrow{QN}_\kappa f$) on $A \subseteq 2^\kappa$ if there exists an unbounded non-decreasing sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that

$$\forall_{x \in A} \exists_{\beta<\kappa} \forall_{\beta \le \alpha < \kappa} f_\alpha(x) \in [f(x) \restriction \xi_\alpha].$$

We obtain the usual implications.

Proposition 41 *If a sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ converges κ-uniformly to a function $f: 2^\kappa \to 2^\kappa$, then $f_\alpha \xrightarrow{QN}_\kappa f$.*

Proposition 42 *If a sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ converges κ-quasi-normally to a function $f: 2^\kappa \to 2^\kappa$, then $f_\alpha \to_\kappa f$.*

We also give examples of sequences of functions which separate those notions. Similarly to the standard case we get the following fact.

Proposition 43 *Let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence of functions $2^\kappa \to 2^\kappa$, and $f: 2^\kappa \to 2^\kappa$. The following conditions are equivalent:*

(1) $f_\alpha \xrightarrow{QN}_\kappa f$ on $A \subseteq 2^\kappa$,

(2) there exists a sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, and for all $\beta < \kappa$, $f_\alpha \rightrightarrows_\kappa f$ on A_β,

(3) there exists a sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, $A_\alpha \subseteq A_\beta$ for all $\alpha < \beta < \kappa$, $\bigcup_{\alpha<\beta} A_\alpha = A_\beta$ for limit $\beta < \kappa$, and for all $\beta < \kappa$, $f_\alpha \rightrightarrows_\kappa f$ on A_β.

On the other hand, we prove the following.

Proposition 44 *Let $A \subseteq 2^\kappa$, and $A = \bigcup_{\alpha<\lambda} A_\alpha$ for $\lambda < \mathfrak{b}_\kappa$. If a sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $A \to 2^\kappa$ converges κ-quasi normally on A_α to $f: A \to 2^\kappa$, for all $\alpha < \lambda$, then $f_\alpha \xrightarrow{QN}_\kappa f$ on A.*

Finally, we prove that a κ-uniform convergent sequence of continuous functions converges to a continuous function.

Proposition 45 *Let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence of continuous functions $2^\kappa \to 2^\kappa$, and $A \subseteq 2^\kappa$. Assume that $f_\alpha \rightrightarrows_\kappa f$ on A, where $f \in A \to 2^\kappa$. Then f is continuous on A.*

We also study special subsets of 2^κ related to convergence of sequences of functions, i.e. κ-QN-sets, κ-wQN-sets and κ-mQN-sets.

A set $A \subseteq 2^\kappa$ is a **κ-QN-set,** if any sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of continuous functions $A \to 2^\kappa$ such that $f_\alpha \to_\kappa \mathbf{0}$ on A, converges also κ-quasi-normally ($f_\alpha \xrightarrow{QN}_\kappa \mathbf{0}$ on A).

A set $A \subseteq 2^\kappa$ is a **κ-weak QN-set (κ-wQN-set),** if for any sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of continuous functions $A \to 2^\kappa$ such that $f_\alpha \to_\kappa \mathbf{0}$ on A, there exists an increasing sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $f_{\xi_\alpha} \xrightarrow{QN}_\kappa \mathbf{0}$ on A.

A set $A \subseteq 2^\kappa$ is a **κ-mQN-set,** if any sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of continuous functions $A \to 2^\kappa$ such that $f_\alpha \to_\kappa \mathbf{0}$ on A, and for all $x \in A$, and $\alpha < \beta < \kappa$

$$\bigcup \{\gamma < \kappa : \forall_{\delta<\gamma} f_\alpha(x)(\delta) = 0\} \leq \bigcup \{\gamma < \kappa : \forall_{\delta<\gamma} f_\beta(x)(\delta) = 0\},$$

converges also κ-quasi-normally ($f_\alpha \xrightarrow{QN}_\kappa \mathbf{0}$ on A).

I present some basic properties of such sets, and we also prove that every κ-wQN-set is κ-perfectly κ-meagre.

Proposition 46 *If $X \subseteq 2^\kappa$ is a κ-wQN-set, then X is κ-perfectly κ-meagre.*

We characterize κ-wQN-sets and κ-QN-sets in terms of κ-sequence selection properties.

Theorem 47 *The following conditions are equivalent.*

(1) X is a κ-wQN-set,

(2) $C_p^\kappa(X)$ has κ-sequence selection property,

(3) $C_p^\kappa(X)$ has $\kappa-(\alpha_2)$ property,

(4) $C_p^\kappa(X)$ has $\kappa-(\alpha_3)$ property,

(5) $C_p^\kappa(X)$ has $\kappa-(\alpha_4)$ property.

Theorem 48 *The following conditions are equivalent.*

(1) X is a κ-QN-set,

(2) $C_p^\kappa(X)$ has $\kappa-(\alpha_1)$ property.

In the above theorems κ-(α_1) – κ-(α_4) are the following properties. If for every $\langle f_{\alpha,\beta}\rangle_{\alpha,\beta<\kappa} \in (C_p^\kappa(X))^{\kappa\times\kappa}$ such that for all $\alpha < \kappa$, $\langle f_{\alpha,\beta}\rangle_{\beta<\kappa}$ converges κ-pointwisely to $\mathbf{0}$, then there exists $\langle g_\alpha\rangle_{\alpha<\kappa} \in (C_p^\kappa(X))^\kappa$ such that $g_\alpha \to_I \mathbf{0}$, and

- for all $\alpha < \kappa$,
$$|\{f_{\alpha,\beta}: \beta \in \kappa\} \setminus \{g_\beta: \beta < \kappa\}| < \kappa,$$
then $C_p^\kappa(X)$ has **property** $\kappa - (\alpha_1)$,
- for all $\alpha < \kappa$,
$$|\{f_{\alpha,\beta}: \beta \in \kappa\} \cap \{g_\beta: \beta < \kappa\}| = \kappa,$$
then $C_p^\kappa(X)$ has **property** $\kappa - (\alpha_2)$,
- $$|\{\alpha < \kappa: |\{f_{\alpha,\beta}: \beta \in \kappa\} \cap \{g_\beta: \beta < \kappa\}| = \kappa\}| = \kappa,$$
then $C_p^\kappa(X)$ has **property** $\kappa - (\alpha_3)$,
- $$|\{\alpha < \kappa: \{f_{\alpha,\beta}: \beta \in \kappa\} \cap \{g_\beta: \beta < \kappa\} \neq \varnothing\}| = \kappa,$$
then $C_p^\kappa(X)$ **has property** $\kappa - (\alpha_4)$.

We also say that $C_p^\kappa(X)$ has **κ-sequence selection property** if for every $\langle f_{\alpha,\beta}\rangle_{\alpha,\beta<\kappa} \in (C_p^\kappa(X))^{\kappa\times\kappa}$ such that $\langle f_{\alpha,\beta}\rangle_{\beta<\kappa}$ converges κ-pointwise to $\mathbf{0}$ for all $\alpha < \kappa$, there exist $\langle\xi_\alpha\rangle_{\alpha<\kappa}, \langle\delta_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $f_{\xi_\alpha,\delta_\alpha} \to_\kappa \mathbf{0}$.

The relation between those classes of special subsets and cover selection principles will be a subject of the further research.

Further on, in Chapter 7, we study the notions of κ-I-convergence and κ-I^*-convergence of sequences of points of 2^κ for an ideal I on κ.

If I is an ideal on κ, then we say that a sequence $\langle x_\alpha\rangle \in (2^\kappa)^\kappa$ **κ-converges to a point** $x \in 2^\kappa$ **with respect to the ideal** I ($x_\alpha \to_{\kappa-I} x$) if for any $\beta < \kappa$

$$\{\alpha < \kappa: x_\alpha \notin [x\restriction\beta]\} \in I.$$

Similarly, a sequence $\langle x_\alpha \rangle \in (2^\kappa)^\kappa$ **κ-I^*-converges to a point** $x \in 2^\kappa$ ($x_\alpha \to_{\kappa-I^*} x$) if there exists $B \in I$ such that $x_{\eta_\alpha} \to_I x$, where $\{\eta_\alpha : \alpha < \kappa\} = \kappa \setminus B$ is the increasing enumeration.

An ideal I on κ is **κ-generated** if there exists a sequence $\langle C_\alpha \rangle_{\alpha<\kappa}$ of elements of I such that for every $A \in I$, there exists $\alpha < \kappa$ such that $A \subseteq C_\alpha$.

If $\lambda \leq \kappa$, we say that an ideal I on κ is **λ-complete** if for any $\mu < \lambda$, and $\mathcal{A} \in [I]^\mu$, $\bigcup \mathcal{A} \in I$.

We start by proving some simple properties. Obviously I^*-convergence implies I-convergence, but this implication can be reversed if I is a κ-P-ideal.

Proposition 49 *Let I be a κ-complete ideal on κ. The following two properties are equivalent.*

(1) For every sequence $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$, and $x \in 2^\kappa$, $x_\alpha \to_{\kappa-I} x$ if and only if $x_\alpha \to_{\kappa-I^*} x$.

(2) I is a κ-P-ideal.

Where **κ-P-ideal** is an ideal which has the following properties.

Proposition 50 *Let I be an ideal on κ. The following statements are equivalent.*

(1) For any sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in I^\kappa$, there exists $B \in I$ such that for every $\alpha < \kappa$, $|A_\alpha \setminus B| < \kappa$.

(2) For any sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in I^\kappa$, there exists a sequence $\langle B_\alpha \rangle_{\alpha<\kappa}$ such that $|A_\alpha \triangle B_\alpha| < \kappa$ for all $\alpha < \kappa$, and $\bigcup_{\alpha<\kappa} B_\alpha \in I$.

Proposition 51 *If I is a κ-complete ideal on κ, then it is a κ-P-ideal if and only if*

(3) For any sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in I^\kappa$ of mutually disjoint sets, there exists a sequence $\langle B_\alpha \rangle_{\alpha<\kappa}$ such that $|A_\alpha \triangle B_\alpha| < \kappa$ for all $\alpha < \kappa$, and $\bigcup_{\alpha<\kappa} B_\alpha \in I$.

Finally, we study properties related to κ-I-Cauchy property.

The notions of κ-I-convergence and κ-I^*-convergence of points of 2^κ allow us to study different notions of ideal convergence of functions $2^\kappa \to 2^\kappa$.

A sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ **converges with respect to an ideal I on κ on a set** $A \subseteq 2^\kappa$:

κ-pointwise ideal, $f_\alpha \to_{\kappa-I} f$ if and only if

$$\forall_{\xi<\kappa} \forall_{x\in A} \{\alpha < \kappa : f_\alpha(x) \notin [f(x) \restriction \xi]\} \in I,$$

κ-quasi-normal ideal, $f_\alpha \xrightarrow{QN}_{\kappa-I} f$ if and only if there exists a sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ which is κ-I-unbounded and

$$\forall_{x\in A} \{\alpha < \kappa : f_\alpha(x) \notin [f(x) \restriction \xi_\alpha]\} \in I,$$

κ-uniform ideal, $f_\alpha \rightrightarrows_{\kappa-I} f$ if and only if

$$\forall_{\xi\in\kappa} \exists_{B\in I} \forall_{x\in A} \{\alpha < \kappa : f_\alpha(x) \notin [f(x) \restriction \xi]\} \subseteq B.$$

κ-I^*-pointwise, $f_\alpha \to_{\kappa-I^*} f$ if and only if for all $x \in A$, there exists $M = \{m_\alpha : \alpha < \kappa\} \subseteq \kappa$, $m_\beta \geq m_\alpha$ for all $\alpha < \beta < \kappa$ such that $\kappa \setminus M \in I$, and $f_{m_\alpha}(x) \to_\kappa f(x)$ on A,

κ-I^*-quasi-normal, $f_\alpha \xrightarrow{QN}_{\kappa-I^*} f$ if and only if there exists $M = \{m_\alpha : \alpha < \kappa\} \subseteq \kappa$, $m_\beta \geq m_\alpha$ for all $\alpha < \beta < \kappa$ such that $\kappa \setminus M \in I$, and $f_{m_\alpha} \xrightarrow{QN}_\kappa f$ on A,

κ-I^*-uniform, $f_\alpha \rightrightarrows_{\kappa-I^*} f$ if and only if there exists $M = \{m_\alpha : \alpha < \kappa\} \subseteq \kappa$, $m_\beta \geq m_\alpha$ for all $\alpha < \beta < \kappa$ such that $\kappa \setminus M \in I$, and $f_{m_\alpha} \rightrightarrows_\kappa f$ on A.

In the above definitions, a sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ is said to be **κ-I-unbounded** if for any $\delta < \kappa$,

$$\{\alpha < \kappa : \xi_\alpha < \delta\} \in I.$$

In particular, κ-I-uniform convergence implies κ-I-quasi-normal convergence, which itself implies κ-I-pointwise convergence. Similarly, κ-I^*-uniform convergence implies κ-I^*-quasi-normal convergence, which itself implies κ-I^*-pointwise convergence. All those implications cannot be reversed.

We also prove the following properties of the above notions of convergence.

Proposition 52 *Let I be an κ-complete ideal on κ. If $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ is such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, $A_\alpha \subseteq A_\beta$ for all $\alpha < \beta < \kappa$, $\bigcup_{\alpha<\beta} A_\alpha = A_\beta$ for limit $\beta < \kappa$, and for all $\beta < \kappa$, $f_\alpha \rightrightarrows_{\kappa-I} f$ on A_β, then $f_\alpha \xrightarrow{QN}_{\kappa-I} f$ on the whole A.*

This implication can be reversed for κ-generated ideals.

Proposition 53 *Let I be a κ-generated, κ-complete ideal on κ. If $f_\alpha \xrightarrow{QN}_{\kappa-I} f$ on $A \subseteq 2^\kappa$, then there exists $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ such that $A = \bigcup_{\alpha<\kappa} A_\alpha$ such that for all $\beta < \kappa$, $f_\alpha \rightrightarrows_{\kappa-I} f$ on A_β.*

Similarly, we have the following proposition.

Proposition 54 *Let I be a κ-complete ideal on κ. If $f_\alpha \xrightarrow{QN}_{\kappa-I^*} f$ on $A \subseteq 2^\kappa$, then there exists $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ such that $A = \bigcup_{\alpha<\kappa} A_\alpha$ and such that for all $\alpha < \kappa$, $f_\alpha \rightrightarrows_{\kappa-I^*} f$ on A_α.*

This implication can be reversed not only for κ-P-ideals.

Proposition 55 *If I is a κ-admissible κ-P-ideal on κ and $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ is such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, $A_\alpha \subseteq A_\beta$ for all $\alpha < \beta < \kappa$, $\bigcup_{\alpha<\beta} A_\alpha = A_\beta$ for limit $\beta < \kappa$, and for all $\alpha < \kappa$, $f_\alpha \rightrightarrows_{\kappa-I^*} f$ on A_α, then $f_\alpha \xrightarrow{QN}_{\kappa-I^*} f$ on the whole A.*

Finally, we have proven that continuity of functions is preserved under uniform limits with respect to an ideal.

Proposition 56 *Let I be an ideal on κ, and let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence of continuous functions $2^\kappa \to 2^\kappa$, and $A \subseteq 2^\kappa$. Assume that $f_\alpha \rightrightarrows_{\kappa-I} f$ on A, where $f \in A \to 2^\kappa$. Then f is continuous on A.*

Proposition 57 *Let I be an ideal on κ, and let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence of continuous functions $2^\kappa \to 2^\kappa$, and $A \subseteq 2^\kappa$. Assume that $f_\alpha \rightrightarrows_{\kappa-I^*} f$ on A, where $f \in A \to 2^\kappa$. Then f is continuous on A.*

We also consider κ-(I,J)-QN-sets and κ-(I,J)-wQN-sets and prove some of their basic properties. Nevertheless, those classes of sets constitute an important subject of future research.

In the final chapter (Chapter 8) we study the possibility of introducing Egorov's Theorem in 2^κ. To achieve this we need a measure analogue in 2^κ. Since no satisfactory concept in known, we define a notion of κ-proto-measure satisfying certain properties which suffice to prove an analogue of Egorov's Theorem.

A triple $\langle \mathbb{L}, \mu, L\rangle$ will be called a **κ-proto-measure** if

(1) $\langle \mathbb{L}, \le\rangle$ is a linear order with the least element.

(2) $\mu: \mathcal{B}_\kappa \to \mathbb{L}$ is a function defined on the family of κ-Borel subsets of 2^κ with values in $\mathbb{L}$.

(3) If $\langle A_\alpha\rangle_{\alpha<\kappa} \in (\mathcal{B}_\kappa)^\kappa$ is such that $\bigcap_{\alpha<\kappa} A_\alpha = \varnothing$, and for all $\alpha < \alpha' < \kappa$, $A_{\alpha'} \subseteq A_\alpha$, then for all $\xi \in \mathbb{L} \setminus \{\min \mathbb{L}\}$, there exists $\delta \in \kappa$ such that $\mu(A_\delta) < \xi$,

(4) $L: (\mathbb{L} \setminus \{\min \mathbb{L}\}) \times \kappa \to \mathbb{L} \setminus \{\min \mathbb{L}\}$,

(5) for all $\xi \in \mathbb{L} \setminus \{\min \mathbb{L}\}$ if $\langle A_\alpha\rangle_{\alpha<\kappa} \in (B_\kappa)^\kappa$ is such that $\mu(A_\alpha) \le L(\xi, \alpha)$ for all $\alpha < \kappa$, then
$$\mu\left(\bigcup_{\alpha<\kappa} A_\alpha\right) \le \xi.$$

We get the following theorems.

Theorem 58 *Let $\langle \mathbb{L}, \mu, L\rangle$ be a κ-proto-measure, and let $\langle f_\alpha\rangle_{\alpha<\kappa}$ be a sequence of κ-measurable functions $2^\kappa \to 2^\kappa$ which are κ-pointwise convergent on κ-Borel set $X \subseteq 2^\kappa$ to $\mathbf{0}$, and let $\xi \in \mathbb{L} \setminus \{\min \mathbb{L}\}$. Then there exists a set $A \subseteq X$ with $A \in \mathcal{B}_\kappa$ and $\mu(X \setminus A) \le \xi$ such that the sequence converges κ-uniformly on A.*

Theorem 59 *Assume that I is a κ-generated κ-complete ideal on κ, and $\langle \mathbb{L}, \mu, L\rangle$ is a κ-proto-measure. Let $\langle f_\alpha\rangle_{\alpha<\kappa}$ be a sequence of κ-measurable functions $2^\kappa \to 2^\kappa$ which are κ-I-pointwise convergent on 2^κ to $\mathbf{0}$, and let $\xi \in \mathbb{L} \setminus \{\min \mathbb{L}\}$. Then there exists a set $A \in \mathcal{B}_\kappa$ with $\mu(2^\kappa \setminus A) \le \xi$ such that the sequence converges κ-I-uniformly on A.*

Theorem 60 *Assume that I is a κ-generated κ-admissible ideal on κ, and $\langle \mathbb{L}, \mu, L\rangle$ is a κ-proto-measure. Let $\langle f_\alpha\rangle_{\alpha<\kappa}$ be a sequence of κ-measurable functions $2^\kappa \to 2^\kappa$ which are κ-I^*-pointwise convergent on 2^κ to $\mathbf{0}$, and let $\xi \in \mathbb{L} \setminus \{\min \mathbb{L}\}$. Then there exists a set $A \in \mathcal{B}_\kappa$ with $\mu(2^\kappa \setminus A) \le \xi$ such that the sequence converges κ-I^*-uniformly on A.*

A κ-proto-measure $\langle \mathbb{L}, \mu, L\rangle$ is **diffused** if for every $x \in 2^\kappa$, $\mu(\{x\}) = \min \mathbb{L}$. It is **increasing** if for every $A, B \in \mathcal{B}_\kappa$ such that $A \subseteq B$, $\mu(A) \le \mu(B)$. Finally, it is **strictly positive** if for every $s \in 2^{<\kappa}$, $\mu([s]) > \min \mathbb{L}$.

A set $A \subseteq 2^\kappa$ is **μ-null** if there exists $B \in \mathcal{B}_\kappa$ such that $A \subseteq B$ and $\mu(B) = \min \mathbb{L}$. The collection of all μ-null subsets of 2^κ is denoted by $\mathcal{N}_\mu$. A set $A \subseteq 2^\kappa$ is **μ-measurable** if there exists $B \in \mathcal{B}_\kappa$ such that $A \triangle B$ is μ-null.

If $\lambda \le 2^\kappa$ is a cardinal, then a κ-proto-measure $\langle \mathbb{L}, \mu, L\rangle$ is **λ-null-complete** if for every $\beta < \lambda$, and sequence $\langle A_\alpha\rangle_{\alpha<\beta}$ of μ-null sets, $\bigcup_{\alpha<\beta} A_\alpha$ is μ-null as well. It is **null-good** if for every $A, B \in \mathcal{B}_\kappa$ if A is μ-null, $\mu(A \cup B) = \mu(B)$.

A κ-proto-measure $\langle \mathbb{L}, \mu, L \rangle$ is **basically transition-invariant** if for any $\alpha < \kappa$, and $t, s \in 2^\alpha$, $\mu([s]) = \mu([t])$.

We also discuss some properties of κ-proto-measures, and prove that every κ-strongly null set is μ-null under some additional assumptions.

Proposition 61 *Assume that $\langle \mathbb{L}, \mu, L \rangle$ is an increasing, diffused, null-good, κ^+-null complete, basically translation-invariant κ-proto-measure. Then every κ-strongly null set is μ-null.*

Proposition 62 *Let κ be a weakly compact cardinal. Assume that $\langle \mathbb{L}, \mu, L \rangle$ is an increasing, diffused, null-good κ-proto-measure. Then every κ-strongly null set is μ-null.*

Although, some simple κ-proto-measures exist, I have not been able to find a κ-proto-measure which is more complex. I do not know whether there exists a non-trivial κ-proto-measure $\langle \mathbb{L}, \mu, L \rangle$ which is

(a) increasing, diffused and null-good.

(b) increasing, diffused and κ-null complete.

(c) diffused and such that $\mathbb{L} = \mathbb{R}_\kappa$ (where $\mathbb{R}_\kappa$ is the Sikorski-Klaua structure of generalized reals ([74], [75], [30], [31], [32], [12] and [10]), and such that for every limit ordinal $\beta < \kappa$, and any sequence $\langle A_\alpha \rangle_{\alpha<\beta} \in (\mathcal{B}_\kappa)^\beta$ such that for $\alpha < \alpha' < \beta$, $A_\alpha \subseteq A_{\alpha'}$, we have

$$\mu\left(\bigcup_{\alpha<\beta} A_\alpha\right) = \sup\{\mu(A_\alpha) \colon \alpha < \beta\}.$$

(d) diffused and such that for every bounded $A \subseteq \mathbb{L}$, there exists $\sup A \in \mathbb{L}$, and such that for every limit ordinal $\beta < \kappa$, and any sequence $\langle A_\alpha \rangle_{\alpha<\beta} \in (\mathcal{B}_\kappa)^\beta$ such that for $\alpha < \alpha' < \beta$, $A_\alpha \subseteq A_{\alpha'}$, we have

$$\mu\left(\bigcup_{\alpha<\beta} A_\alpha\right) = \sup\{\mu(A_\alpha) \colon \alpha < \beta\}.$$

(e) increasing, diffused, and basically transition-invariant.

The existence of such κ-proto-measure is important in the light of proven theorems.

To sum up, it is possible to study theory of special subsets and convergence in 2^κ, although one has to make additional assumptions very often or define notions which are more abstract or intricate than their classical counterparts. Therefore, there is still a wide range of possibilities for the subject of further research in this subject, and my thesis, I hope, lays the groundwork in those cases.

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