Singularities of harmonic and biharmonic maps into compact manifolds

PhD Thesis Extended Abstract

Katarzyna Ewa Mazowiecka

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This thesis is concerned with a study of singular points of *minimizing* harmonic and biharmonic maps.

The standard Dirichlet problem for the Laplace equation is to find a function $u : \Omega \to \mathbb{R}$ defined on a bounded, smooth domain $\Omega \subset \mathbb{R}^m$, which satisfies the equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{0.1}$$

It is well known that for a given $\varphi \in C^0(\partial\Omega)$, there exists a unique solution u such that $u \in C^\infty$. Moreover, any solution to (0.1) is a minimizer of the Dirichlet integral

$$E(u) = \int_\Omega |\nabla u|^2 dx$$

in the class of functions $u : \Omega \to \mathbb{R}$ satisfying the boundary condition $u = \varphi$ on $\partial\Omega$. The latter fact is known as the Dirichlet principle (and was in fact proved in 1940 by H. Weyl, for more details on the historical perspective of the Laplace equation see [3]).

One of the possible ways to generalize this concept is the notion of harmonic maps.

Let $\mathcal{N}$ be a smooth, compact Riemannian manifold without boundary of dimension n. According to J. Nash's embedding theorem [12], we may assume that $\mathcal{N}$ is isometrically embedded in some Euclidean space $\mathbb{R}^\ell$ for ℓ sufficiently large. For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ we define the Sobolev spaces

$$W^{k,p}(\Omega, \mathcal{N}) = \left\{ u \in W^{k,p}(\Omega, \mathbb{R}^\ell) \colon u(x) \in \mathcal{N} \text{ for a.e. } x \in \Omega \right\},$$

equipped with the topology inherited from the topology of the linear Sobolev space $W^{k,p}(\Omega, \mathbb{R}^\ell)$.

Harmonic maps. We focus on the study of singularities of maps $u \colon \mathbb{B}^3 \to \mathbb{S}^2$ which minimize the Dirichlet integral

$$E(u) = \int_{\mathbb{B}^3} |\nabla u|^2 dx, \qquad u \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2) \tag{0.2}$$

under a prescribed boundary condition $u|_{\partial\mathbb{B}^3} = \varphi\colon \mathbb{S}^2 \to \mathbb{S}^2$. Here, $\mathbb{B}^3$ stands for the open unit ball in $\mathbb{R}^3$, $\mathbb{S}^2$ is the unit sphere, and

$$W^{1,2}(\mathbb{B}^3, \mathbb{S}^2) = \{v = (v_1, v_2, v_3) \in W^{1,2}(\mathbb{B}^3, \mathbb{R}^3)\colon |v(x)| = 1 \text{ for a.e. } x \in \mathbb{B}^3\}\,.$$

Moreover, for a map φ in the fractional Sobolev space $H^{1/2}(\mathbb{S}^2, \mathbb{S}^2)$ we write

$$W^{1,2}_{\varphi}(\mathbb{B}^3, \mathbb{S}^2) = \{v \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)\colon v|_{\partial\mathbb{B}^3} = \varphi \text{ in the trace sense}\}\,.$$

Minimizers of the Dirichlet integral (0.2) in $W^{1,2}_{\varphi}(\mathbb{B}^3, \mathbb{S}^2)$ satisfy the Euler–Lagrange system

$$\begin{cases} -\Delta u &= |\nabla u|^2 u \qquad \text{in } \mathbb{B}^3, \\ u|_{\partial\mathbb{B}^3} &= \varphi\,. \end{cases} \tag{0.3}$$

Our main motivation behind this work was to reach a deeper understanding of the mechanisms governing the onset of singularities of solutions, and the cardinality and structure of the set of minimizing solutions for a fixed boundary condition. We also wanted to know whether the *Lavrentiev gap phenomenon*, cf. (0.4) below, occurs typically (in a precise topological meaning). Despite the work of numerous experts over the last three decades, this topic is still not fully understood. Our main result states, roughly speaking, that even in the case when there is no purely topological reason for the solution of (0.3) to be discontinuous, singularities of u *do* occur under arbitrarily small (in the $W^{1,p}$ sense, for $1 \le p < 2$) perturbations of an *arbitrary* smooth boundary data φ.

Before giving formal statements of the results, let us sketch a broader perspective.

When $\deg\varphi \neq 0$, all solutions of (0.3) in $W^{1,2}_{\varphi}(\mathbb{B}^3, \mathbb{S}^2)$ obviously have singularities, as φ has no continuous extension $u\colon \mathbb{B}^3 \to \mathbb{S}^2$. By a celebrated classic theorem of Schoen and Uhlenbeck [15] the singular set of a *minimizing* solution of (0.3) consists of isolated points. By another theorem of Almgren and Lieb [1], if the boundary condition φ has square integrable derivatives on $\mathbb{S}^2$, then the number of these points does not exceed a constant multiple of the *boundary energy* $\int_{\mathbb{S}^2} |\nabla_T\varphi|^2 d\sigma$. (Non-minimizing solutions can behave in a wild way: Rivière [13] proves that for any non-constant boundary data φ there exists an everywhere discontinuous solution of the harmonic map system (0.3).)

However, even when $\varphi\colon \mathbb{S}^2 \to \mathbb{S}^2$ satisfies $\deg\varphi = 0$ — so that a priori there is no topological obstruction for a map $u \in W^{1,2}_{\varphi}(\mathbb{B}^3, \mathbb{S}^2)$ to be continuous — minimizers of E in $W^{1,2}_{\varphi}(\mathbb{B}^3, \mathbb{S}^2)$ might be singular because this is energetically preferable. Hardt and Lin [6] give an example of a smooth zero degree boundary data $\widetilde{\varphi}\colon \mathbb{S}^2 \to \mathbb{S}^2$ which is $H^{1/2}$-close to a constant map and has the following properties:

(a) Each minimizer v of E in $W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2)$ has at least M singular points (the number M can be prescribed a priori);

(b) The Lavrentiev gap phenomenon holds for E in $W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2)$. By this, we mean the following inequality:

$$\mu(\widetilde{\varphi}) := \min_{W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3,\mathbb{S}^2)} E(u) < \overline{\mu}(\widetilde{\varphi}) := \inf_{W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3,\mathbb{S}^2)\cap C^0(\overline{\mathbb{B}}^3)} E(u). \tag{0.4}$$

An immediate consequence of (0.4) is that $W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3,\mathbb{S}^2)\cap C^0(\overline{\mathbb{B}}^3,\mathbb{S}^2)$ is not dense in $W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3,\mathbb{S}^2)$.

As Bethuel, Brezis and Coron have shown, cf. [2, Theorem 5], for boundary conditions φ of zero degree, the Lavrentiev gap phenomenon is equivalent to the fact that all minimizing harmonic maps in $W^{1,2}_{\varphi}(\mathbb{B}^3,\mathbb{S}^2)$ have singularities. Other examples of unexpected and counterintuitive behavior of singularities of minimizing harmonic maps have been given by Almgren and Lieb in [1]. In particular, a minimizer u of E in $W^{1,2}_{\varphi}(\mathbb{B}^3,\mathbb{S}^2)$ can have a large number of singular points even if $\det \nabla_T\varphi \equiv 0$ on $\mathbb{S}^2$ and φ maps the whole sphere $\mathbb{S}^2$ to a smooth curve γ. The abstract of [1] ends with the phrase: "in particular, singularities in u can be unstable under small perturbations of φ."

Our main result ascertains that the message of the last sentence, *singularities can be unstable*, may be strengthened, i. e., replaced with a firm *singularities* are *unstable*, at least when one takes into account small perturbations of the boundary data in the topology of each of the spaces $W^{1,p}$, $1\le p<2$. Here is the precise statement published in [11].

Theorem 1. *Assume that $\varphi\in C^\infty(\mathbb{S}^2,\mathbb{S}^2)$ is an arbitrary smooth map with $\deg\varphi=0$ and $1\le p<2$. Then, for each $\varepsilon>0$ and each $M\in\mathbb{N}$ there exists a map $\widetilde{\varphi}\in C^\infty(\mathbb{S}^2,\mathbb{S}^2)$ such that*

(i) $\deg\widetilde{\varphi}=0$;

(ii) $\|\varphi-\widetilde{\varphi}\|_{W^{1,p}}<\varepsilon$ *and* $\mathcal{H}^2\big(\{x\in\mathbb{S}^2\colon \varphi(x)\neq\widetilde{\varphi}(x)\}\big)<\varepsilon$;

(iii) *the Dirichlet integral E has precisely one minimizer $\widetilde{u}\in W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3,\mathbb{S}^2)$; moreover, $\widetilde{u}$ has at least M point singularities in $\mathbb{B}^3$.*

Combining the above result with Bethuel, Brezis and Coron, [2, Theorems 5–6], one immediately obtains the following.

Corollary 2. *Assume that $\varphi\in C^\infty(\mathbb{S}^2,\mathbb{S}^2)$ and $\deg\varphi=0$. Let $\widetilde{\varphi}\in C^\infty(\mathbb{S}^2,\mathbb{S}^2)$ be given by Theorem 1. Then the Lavrentiev gap phenomenon* (0.4) *holds for $\widetilde{\varphi}$.*

It is a natural question whether the occurrence of such boundary data is a *typical property* in the class of all maps of degree zero, i. e., whether the set of mappings $\widetilde{\varphi}\colon\mathbb{S}^2\to\mathbb{S}^2$ such that conditions (i) and (iii) of Theorem 1 hold *contains a countable intersection of open and dense sets of maps of zero degree* in $H^{1/2}$ (or in some other topology). In spite of some efforts, we have not been able to settle that question.

The main novelty of Theorem 1 and its proof is that (1) we show that the singularities are unstable in a generic sense, (2) in order to achieve that, we show how to combine an appropriately modified idea of Hardt and Lin, applied by them only to *constant* boundary conditions $\phi\colon\mathbb{S}^2\to\{*\}$, with a revisited version of Almgren and Lieb's method of installing new singular points, see [1, Theorem 4.3]. A bridge between these two ingredients is provided by a brief topological argument which guarantees that for each boundary condition φ with $\deg\varphi=0$ there exist two antipodal points $\pm q\in\mathbb{S}^2$ such that φ maps them to the same point of $\mathbb{S}^2$, $\varphi(q)=\varphi(-q)$. We select any pair of such points and, roughly speaking, show how to insert numerous tiny bubbles into φ close to those two antipodal points to obtain the new boundary condition $\tilde{\varphi}$. This way, φ is changed only in two little spherical caps centered at $\pm q \in \mathbb{S}^2$, so that the second statement in (ii) in Theorem 1 does hold.

Finally, we show that the result can be generalized to maps of arbitrary degree.

Theorem 3. *Assume that $\varphi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ is an arbitrary smooth map and $1 \le p < 2$. Then, for each $\varepsilon > 0$ and each $M \in \mathbb{N}$ there exists a map $\widetilde{\varphi} \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ such that*

(i) $\deg \widetilde{\varphi} = \deg \varphi$;

(ii) $\|\varphi - \widetilde{\varphi}\|_{W^{1,p}} < \varepsilon$ *and* $\mathcal{H}^2\big(\{x \in \mathbb{S}^2 \colon \varphi(x) \neq \widetilde{\varphi}(x)\}\big) < \varepsilon$;

(iii) *the Dirichlet integral E has precisely one minimizer $\widetilde{u} \in W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2)$; moreover, $\widetilde{u}$ has at least* $\deg(\varphi) + M$ *point singularities in* $\mathbb{B}^3$.

Biharmonic maps. In the second part of this dissertation we focus on the boundary regularity for *minimizing* biharmonic maps. Our original motivation to study this topic was the desire to understand how, in the model case $u : B^5 \to \mathbb{S}^4$, to modify a boundary map in order to force singularities to appear in the corresponding minimizer of the biharmonic energy. A possible applications of the boundary regularity are wide: We expect that such a result can be used to obtain general nonuniqueness of biharmonic maps as well as examples of nonuniqueness in the class of minimizing biharmonic maps. Furthermore, we suspect a result stating that the boundary data having unique minimizing map are dense in some boundary norms stronger than the natural trace norm.

In the case of second order problems a boundary regularity result for *minimizing* harmonic maps was proved by Schoen and Uhlenbeck [16], for *minimizing* p-harmonic maps[1] by Hardt and Lin [7] and independently by Fuchs [5]. There is also a conditional result for *stationary* harmonic maps [17], which under the assumptions of a boundary monotonicity formula for stationary maps yields a partial regularity at the boundary. See also [14] for a boundary regularity result for another class of harmonic maps.

The main reason for which no partial boundary regularity result is known for *stationary* harmonic maps is the lack of a boundary monotonicity formula. The boundary regularity results for *minimizing* harmonic and p-harmonic maps crucially depend on the existence of a monotonicity formula at the boundary. Such a formula is obtained by reflecting a comparison map used in the proof of a monotonicity formula for minimizing maps, see [16, Lemma 1.3]. A boundary monotonicity formula may be obtained for sufficiently smooth *stationary* harmonic maps. According to [9] such a formula was obtained first by W.Y. Ding, see also [4] and references therein.

Now let us pass to biharmonic setting. The Hessian energy is given by

$$H(u) = \int_\Omega |\Delta u|^2 \, dx.$$

[1] p-harmonic maps are defined similarly as harmonic maps, they are critical points of the p-energy, i.e., $E_p(u) = \int_\Omega |\nabla u|^p \, dx$ among maps in $W^{1,p}(\Omega, \mathcal{N})$.

A map $u \in W^{2,2}(\Omega, \mathcal{N})$ is called *minimizing biharmonic* if, for all maps $v \in W^{2,2}(\Omega, \mathcal{N})$ satisfying $u - v \in W_0^{2,2}$, it holds

$$H(u) \leq H(v).$$

We will be interested in the boundary regularity of minimizing biharmonic maps, so we assume that u satisfies the Dirichlet boundary condition. More precisely, let $\varphi \in C^\infty(\Omega_\delta, \mathcal{N})$ be given for a $\delta > 0$, where

$$\Omega_\delta = \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) < \delta\}.$$

We assume that u satisfies

$$\left(u, \frac{\partial u}{\partial \nu} \right) \Bigg|_{\partial\Omega} = \left(\varphi, \frac{\partial \varphi}{\partial \nu} \right) \Bigg|_{\partial\Omega}, \tag{0.5}$$

where ν denotes the outer normal vector.

Similarly as in the case of harmonic maps a boundary monotonicity formula may be proved for sufficiently smooth biharmonic maps. Gong, Lamm, and Wang gave a biharmonic counterpart of the conditional boundary regularity result for stationary harmonic maps [17] and proved that under the assumption of a boundary monotonicity formula stationary harmonic maps are smooth up to the boundary with off a singular set of codimension 4. Due to the technical difficulties we will not state here the precise formulation of the boundary monotonicity formula.

We show that the conditional partial regularity result of Gong et al. can be strengthen to full boundary regularity in the case of *minimizing* biharmonic maps.

Theorem 4. *Let $m \geq 5$, $\varphi \in C^\infty(\Omega_\delta, \mathcal{N})$ for some $\delta > 0$, assume that $u \in W^{2,2}(\Omega, \mathcal{N})$ is a minimizing biharmonic map, which satisfies the boundary monotonicity inequality. Then, u is smooth on a full neighborhood of the boundary $\partial\Omega$.*

We conjecture that the boundary monotonicity formula is satisfied by all minimizing biharmonic maps with sufficiently smooth boundary data.

Similarly as in the case of harmonic [16] and p-harmonic [7] maps the complete boundary regularity is based on the nonexistence of nonconstant boundary tangent maps. Here is the precise statement

Lemma 5. *Any minimizing biharmonic map $u_0 \in W^{2,2}(B_1^+, \mathcal{N})$ that is homogeneous of degree 0 and that is constant on $B_1 \cap \{x_m = 0\}$ must be a constant.*

We will consider tangent maps at the boundary and prove that they arise as strong limits of rescaled maps on some smaller domain, containing a portion of the boundary. In order to obtain a strong convergence from a sequence we initially only know is uniformly bounded in $W^{2,2}$ we will prove an analogue of Scheven's compactness result.

Scheven, following the result for harmonic maps [9], has based his argument on an analysis of defect measures. We follow his general strategy, modifying numerous technical details so that the proof works for a map obtained via a higher order reflection across a flat portion of the boundary.

We will not prove that a limit u of a weakly convergent sequence of minimizing maps $(u_j)_{j\in\mathbb{Z}}$ is again minimizing. Such a result, is known only in the case when $\mathcal{N} = \mathbb{S}^{\ell-1}$ (see [8, Lemma 3.3.]). In the case of harmonic maps, such a result is known for minimizing maps into arbitrary target manifolds. Since the maps u_j and u slightly differ on the boundary one may not use directly the definition of minimizing map to compare their energies. A tool for comparing those energies was provided by Luckhaus and his lemma in [10]. Unfortunately we may not use directly Luckhaus's lemma to maps from $W^{2,2}$. An analogue of this lemma is not known in the biharmonic setting.

Instead, similarly as in [15, 16] and [7] for us it will be sufficient that in very simple situations a limit of *minimizing* maps is again *minimizing*. By a repeated formation of tangent boundary maps we arrive at a boundary tangent map which has a special form — it is independent of the first $(m-5)$-variables, homogeneous of degree 0, whose only discontinuity may occur at the origin. It was proved by Scheven that such maps are in fact minimizing.

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