The Gysin homomorphism for homogeneous spaces via residues

PhD dissertation summary

1 The Gysin homomorphism in equivariant cohomology

Let $G$ be a topological group and let $BG$ denote the classifying space of $G$ ([May99]). Let $EG \to BG$ be the universal principal $G$-bundle, in particular $EG$ is a contractible space on which $G$ acts freely. For a topological space $X$ with a right $G$-action, one defines the homotopy quotient of $X$ by $G$, denoted by $EG \times^G X$, to be the quotient of $EG \times X$ by the diagonal action of $G$ given by $g(e, x) = (eg^{-1}, gx)$. The homotopy quotient has a natural structure of a fiber bundle $EG \times^G X \to BG$ with fiber $X$, via the map induced by the projection on the first factor. The equivariant cohomology of $X$ with coefficients in a ring $R$ is, by definition, the singular cohomology of the homotopy quotient,

$$\tilde{H}_G^*(X; R) := H^*(EG \times^G X; R).$$

The total space of the universal bundle $EG$ is typically infinite dimensional. While working with algebraic varieties with an action of a linear algebraic group, one often uses finite dimensional approximation spaces $E_m$ with a free $G$-action, whose connectivity diverges to infinity as $m$ goes to infinity. The spaces $E_m$ allow to compute the equivariant cohomology of $X$, in certain range. If $E_m$ is $k(m)$-connected, then for $i < k(m)$ there are natural isomorphisms $H^i(E_m \times^G X) \cong H^i_G(X)$.

The Gysin homomorphism, introduced under the name *Umkehrungshomomorphismus* in [Gys41], originally associated to a map $f : M \to N$ of closed oriented manifolds a "wrong way" homomorphism $f_* : H^*(M) \to H^*(N)$ between cohomology groups and was later generalized to many different settings. The Gysin map for fiber bundles and de Rham cohomology has a natural interpretation as integration along fibers. Gysin maps have become an important tool in algebraic geometry, due to the growing importance of characteristic
Let $f : X \to Y$ be a map of closed oriented manifolds. The **Gysin map**, $f_* : H^*(X) \to H^*(Y)$, also called the **push-forward** in cohomology, is defined by the following diagram, in which the vertical maps are Poincaré duality isomorphisms and the bottom map is the push-forward in homology.

\[
\begin{array}{c}
H^*(X) \xrightarrow{f_*} H^{*-\dim X + \dim Y}(Y) \\
\| P.D. \| \quad \| P.D. \|
\end{array}
\]

H_{\dim X - s}(X) \xrightarrow{f_*} H_{\dim X - s}(Y)

One analogously defines the push-forward of a proper map $f : X \to Y$ of noncompact manifolds, replacing the homology groups with the Borel-Moore homology groups.

In equivariant cohomology, the Gysin maps are defined as the standard Gysin maps applied to the approximation spaces: for a proper morphism $f : X \to Y$ one defines

\[
f^G_* : H^*_G(X) \to H^*_{G^d}(Y),
\]

where $d = \dim Y - \dim X$, as the classical Gysin homomorphism for the map

\[
\mathbb{E}G_m \times^G X \to \mathbb{E}G_m \times^G Y.
\]

In the case when $f : X \to pt$ is the constant map, the Gysin homomorphism $f^G_* : H^*_G(X) \to H^*_G(pt)$ associated to it is often denoted by $a \mapsto \int_X a$.

In [Zie14] we have presented a new approach to push-forwards in equivariant cohomology of homogeneous spaces of classical Lie groups, inspired by the push-forward formula for flag varieties of Bérczi and Szenes in [BS12]. For a homogeneous space $G/P$ of a classical semisimple Lie group $G$ and its maximal parabolic subgroup $P$ with an action of a maximal torus $T$ in $G$ we express the Gysin homomorphism $\pi_* : H^*_T(G/P) \to H^*_T(pt)$ associated to a constant map $\pi : G/P \to pt$ in the form of an iterated residue at infinity of a certain complex variable function. The residue formula depends only on the combinatorial properties of the homogeneous space involved, in particular on the Weyl groups of $G$ and $P$ acting on the set of roots of $G$. The formulas were obtained using localization techniques.

In this dissertation we show how to obtain the formulas of [Zie14] in the context of symplectic reductions, using the Jeffrey–Kirwan nonabelian localization theorem ([JK95]) to provide a natural geometric interpretation for the
mentioned formulas. For a compact Lie group $K$ acting in a Hamiltonian way on a symplectic manifold $\mathcal{M}$, one considers the symplectic reduction $\mathcal{M}/\!/K$ together with the Kirwan map $\kappa : H^*_K(\mathcal{M}) \to H^*(\mathcal{M}/\!/K)$. By a theorem by Kirwan ([Kir84]), $\kappa$ is an epimorphism. Moreover, the push-forwards of images under $\kappa$ of singular cohomology classes can be expressed as residues at infinity of a certain expression. We extend this result to push-forwards to a point in equivariant cohomology. Together with a generalized version of a theorem by Martin ([Mar00]), which relates the cohomology of symplectic reduction by a compact Lie group $K$ and its maximal torus $S$, one can use nonabelian localization to to find push-forward formulas in equivariant cohomology.

2 Equivariant Jeffrey–Kirwan theorem

A smooth manifold $\mathcal{M}$ is called a symplectic manifold if it is equipped with a symplectic form $\omega$, i.e. a closed, nondegenerate, skew-symmetric differential 2-form. Morphisms in the category of symplectic manifolds are the symplectomorphisms, which are smooth maps preserving the symplectic form.

An action of a compact Lie group $K$ on $\mathcal{M}$ is called Hamiltonian, if the fundamental vector fields $v_\xi$ associated to the action satisfy

$$dH_\xi = \iota(v_\xi)\omega,$$

and the choice of functions $H_\xi$ is consistent in the sense that the association $\xi \mapsto H_\xi$ is a homomorphism of Lie algebras $\hat{\mu} : \mathfrak{k} \to C^\infty(\mathcal{M})$. This homomorphism determines a map $\mu : \mathcal{M} \to \mathfrak{k}^*$, called the moment map of the action, by dualization.

If $K$ acts on $\mathcal{M}$ in a Hamiltonian way with moment map $\mu : \mathcal{M} \to \mathfrak{k}^*$ and 0 is a regular value of $\mu$, then the symplectic reduction of $\mathcal{M}$ with respect to $K$ is defined as

$$\mathcal{M}/\!/K := \mu^{-1}(0)/K.$$

The Kirwan map $\kappa$ is a map relating the cohomology of the symplectic reduction with the equivariant cohomology of the unreduced manifold,

$$\kappa : H^*_K(\mathcal{M}) \to H^*(\mu^{-1}(0)/K),$$

and is defined as the following composition:

$$\kappa : H^*_K(\mathcal{M}) \xrightarrow{i^*} H^*_K(\mu^{-1}(0)) \xrightarrow{(\pi^*)^{-1}} H^*(\mu^{-1}(0)/K),$$

where $i^*$ is the map of $K$-equivariant cohomology induced by the inclusion $i : \mu^{-1}(0) \to \mathcal{M}$ and $\pi^*$ is the natural isomorphism induced by the quotient
map \( \pi : \mu^{-1}(0) \to \mu^{-1}(0)/K \).

The classical Jeffrey–Kirwan theorem in a reformulation of Guillemin and Kalkman ([GK96]) gives the following formula for the nonequivariant push-forward to a point in cohomology of \( \mathcal{M}/K \).

**Theorem 1** (Guillemin–Kalkman). Let \( \mathcal{M} \) be a compact symplectic manifold equipped with a Hamiltonian action of a compact Lie group \( K \). Let \( S \) denote the maximal torus in \( K \) and assume the \( S \)-fixed points are isolated. Let \( \mathcal{W} = N_K(S)/S \) denote the Weyl group of \( K \). For a fixed point \( p \), let \( i_p \) denote the inclusion of \( p \) into \( \mathcal{M} \) and let \( \{ \lambda_i(p) \}_{i=1}^{\dim S} \) be the weights of the isotropy representation of \( S \) at \( p \). Finally let \( \varpi = \mathbb{C}[z] \) be the product of the roots of \( K \).

One can choose a subset \( D \) of the fixed points set of the \( S \)-action and for each point \( p \in D \) a basis \( z = (z_1, \ldots, z_n) \) of \( s^* \) such that for \( \alpha \in H^*_K(\mathcal{M}) \) one has:

\[
\int_{\mathcal{M}/K} \kappa(\alpha) = \frac{1}{|\mathcal{W}|} \sum_{p \in D} \text{Res}_{z=\infty} \varpi \cdot i_p^* \alpha \prod_{i=1}^{\dim S} \lambda_i(p).
\]

We extend this theorem to \( T \)-equivariant cohomology, hence obtaining a residue-type formula for the Gysin map in \( T \)-equivariant cohomology of \( \mathcal{M}/K \).

Let \( \mathcal{M} \) be a compact symplectic manifold equipped with a Hamiltonian action of two a torus \( S \) and an action of a torus \( T \), such that the two actions commute. Denote by \( \mu_S \) the moment map for the action of \( S \) and assume \( 0 \) is a regular value of \( \mu_S \). Assume additionally that the set \( \mu_S^{-1}(0) \) is \( T \)-invariant. We define a \( T \)-equivariant analogue of the Kirwan map for the \( S \)-action:

\[
\kappa_T : H^*_T(S) (\mathcal{M}) \xrightarrow{i^*_T} H^*_T(\mu_S^{-1}(0)) \xrightarrow{(q^*)^{-1}} H^*_T(\mu_S^{-1}(0)/S) = H^*_T(\mathcal{M}/S),
\]

which is defined as the composition of the map induced on equivariant cohomology by the inclusion \( i : \mu_S^{-1}(0) \hookrightarrow \mathcal{M} \) (or, equivalently, as the map induced on singular cohomology by the inclusion \( ET \times T \mu_S^{-1}(0) \hookrightarrow ET \times T \mathcal{M} \)) with the inverse of the natural isomorphism \( q^* : H^*_T(\mu_S^{-1}(0)/S) \to H^*_T(S) (\mu_S^{-1}(0)) \).

Let \( \mathcal{M}_{\text{critical}} \) be the set of critical points of the moment map \( \mu_S \). By the results of Guillemin and Sternberg [GS82] it admits a decomposition into a finite union

\[
\mathcal{M}_{\text{critical}} = \bigcup_j \mathcal{M}_j,
\]

where each \( \mathcal{M}_j \) is a fixed point set of a one-dimensional subgroup \( S_j \) of \( S \). Consider the equivariant Kirwan map for the action of \( H_j = S/S_j \) on \( \mathcal{M}_j \)

\[
\kappa^j_T : H^*_T(\mathcal{M}_j) \to H^*_T(\mathcal{M}_j//H_j),
\]
and let \( \kappa_T \) be the \( T \)-equivariant Kirwan map for the action of \( S \) on \( \mathcal{M} \). Let \( i_j \) be the inclusion \( \mathcal{M}_j \to \mathcal{M} \) and let \( e^{T \times S}(\nu_j) \) be the equivariant Euler class of the normal bundle \( \nu_j \) to \( \mathcal{M}_j \) in \( \mathcal{M} \). We choose a generator \( x_j \) of \( H^*_S(pt) \). Define

\[
\text{res}_j(\alpha) := \text{res}_{x_j} = \infty \frac{i_j^* \alpha}{e^{T \times S}(\nu_j)}.
\]

**Theorem 2** (Equivariant Guillemin–Kalkman Theorem). Let \( \alpha \) be a cohomology class in \( H^*_T(\mathcal{M}) \). One can choose a subset \( \mathcal{F} \) of connected components of the fixed point set \( \mathcal{M}^S \) and for each \( F \in \mathcal{F} \) a sequence of subtori

\[
S_F^{(1)} \subseteq S_F^{(2)} \subseteq \cdots \subseteq S_F^{(N)} = S,
\]

with \( \dim S_F^{(i)} = i \) and a basis \( x_{F,1}, \ldots, x_{F,n} \) of \( \mathfrak{s}^* \) such that for each \( i \) the dual elements \( x_{F,1}^*, \ldots, x_{F,n}^* \) form a basis of the integer lattice in the Lie algebra \( \mathfrak{s}_F^{(i)} \) of \( S_F^{(i)} \), such that the \( T \)-equivariant push-forward to a point of \( \kappa_T(\alpha) \) is given by the following formula:

\[
\int_{\mathcal{M}/S} \kappa_T(\alpha) = \sum_{F \in \mathcal{F}} \int_F \text{res}_{x_{F,n} = \infty} \ldots \text{res}_{x_{F,1} = \infty} \frac{i_F^* \alpha}{e^{T \times S}(\nu)}.
\]

In the above formula \( e^{T \times S}(\nu) \) denotes the \( T \times S \)-equivariant Euler class of the normal bundle to \( F \) in \( \mathcal{M} \).

The proof of this theorem is the content of Chapter 2. of the dissertation.

### 3 Equivariant Martin integration formula

Let \( \mathcal{M} \) be a compact symplectic manifold equipped with a Hamiltonian action of a compact Lie group \( K \). Let \( S \) be a maximal torus in \( K \), acting on \( \mathcal{M} \) by restriction of the \( K \)-action. Let \( \mu_K : \mathcal{M} \to \mathfrak{k}^* \) be the moment map for the \( K \)-action, and \( \mu_S \) the moment map for the action of \( S \). The following theorem of Martin ([Mar00]) relates the Gysin homomorphisms for the symplectic reductions \( \mathcal{M}\!/K \) and \( \mathcal{M}\!/S \).

**Theorem 3** (Martin Integration Formula). Let \( \mathfrak{W} \) denote the Weyl group of \( K \), acting naturally on \( \mathcal{M}\!/S \). To any weight \( \alpha \) of \( S \) we associate the equivariant line bundle \( L_\alpha \) over \( \mathcal{M}\!/S \) and define \( e(\alpha) := e(L_\alpha) \) to be the Euler class of this bundle. Denote \( e = \prod_{\alpha \in \mathfrak{W}} e(L_\alpha) \). Let \( \mu_K^{-1}(0)/S \to \mathcal{M}\!/S \) be the map induced by inclusion \( \mu_K^{-1}(0) \to \mu_S^{-1}(0) \) and consider the map \( \pi : \mu_K^{-1}(0)/S \to \mathcal{M}\!/K \), which is a fibration with fiber \( K/S \). The class \( \tilde{a} \in H^*(\mathcal{M}\!/S) \) is called a **lift**.
of the class \( a \in H^* (\mathcal{M} \sslash K) \) if \( \pi^* a = i^* \tilde{a} \). Then if \( \tilde{a} \) is a lift of \( a \), the following formula holds

\[
\int_{\mathcal{M} \sslash K} a = \frac{1}{|W|} \int_{\mathcal{M} \sslash S} \tilde{a} \cdot e.
\]

We extend this result to \( T \)-equivariant cohomology as follows.

Let \( \mathcal{M} \) be a symplectic manifold equipped with two commuting actions: a Hamiltonian action of a compact Lie groups \( K \) and an action of a torus \( T \). Let \( S \) be a maximal torus in \( K \), acting by restriction of the action of \( K \), and let 0 be a regular value of the moment map \( \mu_K \). Assume that the symplectic reduction \( \mathcal{M} \sslash S \) is compact. Assume that the sets \( \mu_K^{-1}(0), \mu_S^{-1}(0) \) are \( T \)-invariant. Consider the maps

\[
i : \mu_K^{-1}(0)/S \hookrightarrow \mathcal{M} \sslash S = \mu_S^{-1}(0)/S
\]

which is an inclusion induced by the inclusion \( \mu_K^{-1}(0) \hookrightarrow \mu_S^{-1}(0) \) and

\[
\pi : \mu_K^{-1}(0)/S \to \mathcal{M} \sslash K
\]

which is a fibration with fiber \( K/S \). A cohomology class \( \tilde{\alpha} \in H^*_T (\mathcal{M} \sslash S) \) is called a lift of \( \alpha \in H^*_T (\mathcal{M} \sslash K) \), if \( \pi^* \alpha = i^* \tilde{\alpha} \).

**Theorem 4** (Equivariant Martin Integration Formula). For a weight \( \gamma \) of the \( S \)-action let \( C_\gamma \) denote the vector space \( \mathbb{C} \) with the action of \( S \) given by \( \gamma \). Let \( e^T = \prod_{\gamma \in \Phi} e^T(\gamma) \in H^*_T (\mathcal{M} \sslash S) \) be the product of \( T \)-equivariant Euler classes associated to the roots \( \Phi \) of \( K \), denoted \( e^T(\gamma) := e(L^T_\gamma) \), where \( L^T_\gamma = ET \times^T \mu_S^{-1}(0) \times^S C_\gamma \to \mathcal{M} \sslash S \). Then

\[
\int_{\mathcal{M} \sslash K} \alpha = \frac{1}{|W|} \int_{\mathcal{M} \sslash S} \tilde{\alpha} \cdot e^T,
\]

where the integrals denote push-forwards to a point in \( T \)-equivariant cohomology i.e. \( \int_{\mathcal{M} \sslash K} - := p_*(-) \) for \( p : \mathcal{M} \sslash K \to pt \).

The proof of this result is the content of Chapter 3. of the dissertation.

### 4 Push-forward formulas

Consider the partial flag variety of type \( d = (d_1, \ldots, d_k) \) in \( W \simeq \mathbb{C}^n \)

\[
\text{Fl}_d(W) = \{ V_1 \subset V_2 \subset \cdots \subset V_k \subset W : V_i \text{ linear subspace of } W, \dim V_i = d_i \}. 
\]
One can show (see [Kam]) that Fl_d(W) can be constructed as a symplectic reduction, as follows. Let \{V_1, V_2, \ldots, V_k\} be a collection of Hermitian vector spaces of dimensions \( \dim V_i = d_i \). Define

\[
\text{Hom}(V, W) := \bigoplus_{i=1}^{k-1} \text{Hom}(V_i, V_{i+1}) \oplus \text{Hom}(V_k, W).
\]

This is a symplectic manifold, since each of the Hermitian vector spaces \( \text{Hom}(V_i, V_{i+1}) \) is a symplectic manifold, with the symplectic form given by \( \omega(A, B) = 2 \Im(\text{tr} AB^*) \).

Let \( U(V) := U(V_1) \times \cdots \times U(V_k) \) and consider the action of \( U(V) \) on \( \text{Hom}(V, W) \) given by

\[
(g_1, \ldots, g_k)(A_1, \ldots, A_{k-1}, B) = (g_2 A_1 g_1^{-1}, \ldots, g_k A_{k-1} g_{k-1}^{-1}, B g_k^{-1}).
\]

This action is Hamiltonian and one can choose \( \lambda = (\lambda_1 \text{Id}_{d_1}, \ldots, \lambda_k \text{Id}_{d_k}) \in \mathfrak{u}(V) \) such that there is an isomorphism

\[
\text{Hom}(V, W)_{/\lambda} \cong U(V) \cong \text{Fl}_d(W).
\]

Consider two torus actions on \( \text{Hom}(V, W) \):

- The action of the maximal torus \( S = S_1 \times \cdots \times S_k \) contained in \( U(V) = U(V_1) \times \cdots \times U(V_k) \) acting by the restriction of the above action.
- The action of the \( n \)-dimensional torus \( T \) acting in \( \text{Hom}(V, W) \) by matrix multiplication on the left on the last component \( \text{Hom}(V_k, W) \).

Let us denote the characters of \( T \) by \( t = \{t_1, \ldots, t_n\} \) and the characters of \( S \) by \( z \). The set \( z \) is the union of the sets of characters of the tori \( S_i \), which we denote by \( z_{i,1}, \ldots, z_{i,d_i} \) for \( i = 1, \ldots, k \). One can reduce the Gysin homomorphism for \( \text{Fl}_d(W) \) to the Gysin map for the symplectic reduction of \( \text{Hom}(V, W)_{/S} \) using Theorem 4 and use Theorem 2 to convert it into a residue. Let \( \kappa_T \) be the \( T \)-equivariant Kirwan map for the action of \( U(V) \) on \( \text{Hom}(V, W) \) and let \( \kappa_T^S \) be the \( T \)-equivariant Kirwan map for the action of \( S \) on \( \text{Hom}(V, W) \). Denote by \( e \) the \( T \)-equivariant characteristic class corresponding to the product of roots of \( U(V) \). Then

\[
\int_{\text{Fl}_d(W)} \kappa_T(\alpha) = \frac{1}{|W|} \int_{\text{Hom}(V, W)_{/S}} \kappa_T^S(\alpha) \cdot e
\]

\[
= \frac{1}{|W|} \text{Res}_{w=\infty} \frac{i_0^* \kappa_T^S(\alpha \cdot \hat{e})}{e^{T \times S}(0)},
\]

(1)
for a certain class \( \tilde{e} \) such that \( \kappa^S_T(\tilde{e}) = e \). The residue is taken with respect to a suitably chosen set of variables associated to the weights of the action. From the above expression one can derive the following push-forward formula.

**Theorem 5.** Let \( \alpha \in H^*_T \times K(M)^{2\mathfrak{W}} \), where \( 2\mathfrak{W} = \Sigma_{d_1} \times \cdots \times \Sigma_{d_k} \) is the Weyl group of \( U(V) \) and let \( \{z_1, \ldots, z_{d_k}\} \) denote the characters of the last component of the maximal torus \( S < K = U(V_1) \times \cdots \times U(V_k) \). Then

\[
\int_{\text{Fl}_d(W)} \kappa_T(\alpha) = \frac{1}{|2\mathfrak{W}|} \text{Res}_{z_1,\ldots,z_{d_k}=\infty} \alpha \cdot \prod_{i \neq j, (i,j) \in I_{Fl}} (z_i - z_j) \prod_{l=1}^{d_k} \prod_{m=1}^{n} (-z_l + t_m)
\]

The indexing set \( I_{Fl} \) is depicted on the following diagram (the indices coloured in grey belong to \( I_{Fl} \), the white ones don’t).

One can similarly describe the push-forward formulas for the partial flag varieties of type \( C \) (the isotropic flags in a space \( W \simeq \mathbb{C}^{2n} \) with a given symplectic form) and types \( B \) and \( D \) (the isotropic flags in a space \( W \simeq \mathbb{C}^{2n} \) or \( W \simeq \mathbb{C}^{2n+1} \) with a given symmetric bilinear form). The obtained results are the following.

**Theorem 6.** Let \( \alpha \in H^*_T \times U(V)(\text{Hom}(V,W))^{2\mathfrak{W}} \), where \( T \) is a maximal torus in \( Sp(n) \). Let \( \mathbf{z} = \{z_1, \ldots, z_{d_k}\} \) denote the characters of the last component of the maximal torus \( S < U(V_1) \times \cdots \times U(V_k) \). We get the following expression for the push-forward of a class \( \kappa(\alpha) \).
\[
\int_{F_{\mathfrak{T}}^k(W)} \kappa_T(\alpha) = \frac{1}{|W|} \text{Res}_{z=\infty} \frac{\alpha \cdot \prod_{i \neq j} (z_i - z_j) \prod_{i, j \in I_{F_1}} (z_i + z_j)}{\prod_{l=1}^{n} \prod_{m=1}^{d_k} (t_m - z_l)(t_m + z_l)},
\]

The set \(I_{F_1}\) is the indexing set of Theorem 5.

**Theorem 7.** Let \(\alpha \in H^*_T \times U(V) \otimes \operatorname{Hom}(V, W)\), where \(T\) is a maximal torus in \(SO(2n)\). Let \(z = \{z_1, \ldots, z_{d_k}\}\) denote the characters of the last component of the maximal torus \(S < U(V_1) \times \cdots \times U(V_k)\).

\[
\int_{F_{\mathfrak{T}}^k(W)} \kappa_T(\alpha) = \frac{1}{|W|} \text{Res}_{z=\infty} \frac{2^{d_k} \alpha \cdot \prod_{i \neq j} (z_i - z_j) \prod_{i, j \in I_{F_1}} (z_i + z_j)}{\prod_{l=1}^{n} \prod_{m=1}^{d_k} (t_m - z_l)(t_m + z_l)}.
\]

The set \(I_{F_1}\) is the indexing set of Theorem 5.

**Theorem 8.** Let \(\alpha \in H^*_T \times U(V) \otimes \operatorname{Hom}(V, W)\), where \(T\) is a maximal torus in \(SO(2n + 1)\). Let \(z = \{z_1, \ldots, z_{d_k}\}\) denote the characters of the last component of the maximal torus \(S < U(V_1) \times \cdots \times U(V_k)\).

\[
\int_{F_{\mathfrak{T}}^k(W)} \kappa_T(\alpha) = \frac{1}{|W|} \text{Res}_{z=\infty} \frac{\alpha \cdot \prod_{i \neq j} (z_i - z_j) \prod_{i, j \in I_{F_1}} (z_i + z_j) \prod_{i=1}^{d_k} z_i}{\prod_{l=1}^{n} \prod_{m=1}^{d_k} (t_m - z_l)(t_m + z_l)}.
\]

The set \(I_{F_1}\) is the indexing set of Theorem 5.

The derivation of the push-forward formulas is the content of Chapter 4 of the dissertation.

**References**


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