Our purpose is to give a new constructive method of derivation of Hardy inequalities. We build inequalities knowing solutions $u$ to $p$ and $A$–harmonic problems. We derive Caccioppoli inequalities for $u$. As a consequence we obtain weighted Hardy inequalities for compactly supported Lipschitz functions.

In the first part we obtain one–parameter family of Hardy inequalities of the form

$$
\int_\Omega |\xi(x)|^p \mu_{1,\beta}(dx) \leq \int_\Omega |\nabla \xi(x)|^p \mu_{2,\beta}(dx),
$$

where $1 < p < \infty$, $\xi : \Omega \to \mathbb{R}$ is a compactly supported Lipschitz function, and $\Omega$ is an open subset of $\mathbb{R}^n$ not necessarily bounded. The involved measures $\mu_{1,\beta}(dx), \mu_{2,\beta}(dx)$ depend on certain parameter $\beta$ and $u$ — a nonnegative weak solution to anticoercive PDI

$$
-\Delta_p u \geq \Phi \quad \text{in} \quad \Omega,
$$

with locally integrable function $\Phi$. We allow quite a general function $\Phi$ that can be negative or sign changing if only there exists a finite number

$$
\sigma_0 := \inf \{\sigma \in \mathbb{R} : \Phi \cdot u + \sigma |\nabla u|^p \geq 0 \quad \text{a.e. in} \quad \Omega \cap \{u > 0\}\}.
$$

(1)

The second part is devoted to Hardy inequalities of the form

$$
\int_\Omega F_{\tilde{A}}(|\xi|) \mu_1(dx) \leq \int_\Omega \tilde{A}(|\nabla \xi|) \mu_2(dx),
$$

where $\xi : \Omega \to \mathbb{R}$ is a compactly supported Lipschitz function, $\Omega$ is an open subset of $\mathbb{R}^n$ not necessarily bounded, $\tilde{A}(|\lambda|) = A(\lambda)\lambda$ is an $N$–function satisfying $\Delta'$–condition (i.e. $\tilde{A}(x,y) \leq C_A \tilde{A}(x) \tilde{A}(y)$ for every $x,y > 0$), and $F_{\tilde{A}}(t) = 1/(\tilde{A}(1/t))$. The involved measures $\mu_1(dx), \mu_2(dx)$ depend on $u$ — a nonnegative weak solution to the anticoercive partial differential inequality of elliptic type involving $A$–Laplacian:

$$
-\Delta_A u = -\text{div} A(\nabla u) \geq \Phi \quad \text{in} \quad \Omega,
$$

with locally integrable function $\Phi$, satisfying a variant of Condition (1). The results of the second part imply those of the first part with all details. In particular, the constants which we obtain in both attempts are equal.

Our method of construction of the inequalities is a handy tool. Not only is it easy to conduct but also it gives deep results such as classical inequalities with the best constants. Caccioppoli inequalities for solutions to $p$ and
$A$–harmonic problems may be applied in qualitative and quantitative theory of PDEs. Furthermore, the obtained results reveal connections between nonlinear problems and embedding theorems for linear operators, which builds a bridge between nonlinear and linear analysis.